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Caputo Equations in the frame of fractional operators with Mittag-Leffler kernels

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ABSTRACT : *In this paper, we first use fractional integral formula to study the fractional integral property of ABC and ABR. Secondly, we present integration by parts formulas for the ABC and ABR fractional derivatives with Mittag-Leffler kernels when the operators is.* $\alpha \in (1, \frac{3}{2})$ *. Finally, we investigate the self-adjointness, eigenvalues, and eigenfunction properties of the corresponding fractional Sturm-Liouville equations by using fractional integration by parts formulas.*

KEYWORDS: *Mittag-Leffler kernels; ABR and ABC fractional derivatives; Sturm- Liouville equations.*

I. INTRODUCTION

The fractional differential plays a key role in the development of mathematics and science fields ([4], [7], [8]), especially for the discrete fractional calculus was of interest among several mathematicians and has been developing rapidly. In recent years, it is very meaningful to develop new non-local fractional derivatives ([5], [10]). The proposed kernels are non-singular such as those with *Mittag-Leffler* kernels. What makes those fractional derivatives with *Mittag-Leffler* kernels more interesting is that their corresponding fractional integrals contain *Riemann-Liouville* fractional integrals as a part of their structure. The advantage of develop more efficient algorithms in solving fractional dynamical systems by concentrating only on the differential equations. In this paper, we present integration by parts formulas for the *ABC* and *ABR* fractional differences with *Mittag-Leffler* kernels of order $\alpha \in (1, \frac{3}{2})$ which will be used to prove the self-adjointness, eigenvalues, and eigenfunction properties of suitable fractional difference *Sturm-Liouville* equations with proper boundary conditions. From [13], we know the classical *Sturm-Liouville* problem for a linear differen-tial equation of second order is a boundary value problem of the form:

$$\begin{aligned} &-\frac{d}{dt}(p(t)\frac{dx}{dt}) + q(t)x(t) = \lambda r(t)x(t), \quad t \in [a,b], \\ &c_1 x(a) + c_2 x'(a) = 0, \\ &d_1 x(b) + d_1 x'(b) = 0, \end{aligned}$$

where p, q, r are continuous functions on the interval $[a,b]$ such as $q(t) > 0$,

$r(t) > 0$ on $[a,b]$. The differential equation can be written in the from:

$$L(x) = \lambda r(t)x(t)$$

where $L(x) = {}^cD^\alpha(p(t){}^cD^\alpha x)(t) + q(t)x(t)$. Parameter λ for which the above boundary value problem has a nontrivial solution is called an eigenvalue.

II. PRELIMINARIES

Definition 1. ([2]) The *Mittag-Leffler* function of one parameter α is defined by

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}, \alpha\in C, \mathrm{Re}(\alpha)>0$$

and the *Mittag-Leffler* function of two parameters α,β is defined by

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \alpha,\beta\in C, \mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0$$

where $E_{\alpha,1}(z)=E_\alpha(z)$.

Definition 2. ([13]) The left fractional integral of order $\alpha>0$ starting at *a* has the following from

$$({}_aI^\alpha f)(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}f(s)ds$$

The right fractional integral of order $\alpha>0$ ending at *b* has the following from

$$(I_b^\alpha f)(t)=\frac{1}{\Gamma(\alpha)}\int_t^b(s-t)^{\alpha-1}f(s)ds,$$

where $\alpha>0, t\in[a,b], f$ are continuous on the interval $[a,b]$.

Definition 3. ([5]) Let $f\in H^1(a,b), \alpha\in(1,\frac{3}{2}), \beta=\alpha-1$,Then the left *Caputo* fractio-

nal derivative is defined by

$$({}^{ABC}_{a}D^\alpha f)(t)=({}^{ABC}_{a}D^\beta f')(t). \quad (1)$$

Then the left *Riemann-Liouville* fractional derivative is defined by

$$({}^{ABR}_{a}D^\alpha f)(t)=({}^{ABR}_{a}D^\beta f')(t). \quad (2)$$

In addition, the corresponding fractional integral is defined by

$$({}^{AB}_{a}I^\alpha f)(t)=({}_aI\,{}^{AB}_{a}I^\beta f)(t),\ ({}^{AB}_{a}I^\alpha f)(t)=({}_aI^{\beta+1}f)(t) \quad (3)$$

Let $f\in H^1(a,b),\ \alpha\in(1,\frac{3}{2}),\ \beta=\alpha-1$, Then the right *Caputo* fractional derivative is defined by

$$({}^{ABC}D_b^{\ \alpha} f)(t)=-({}^{ABC}D_b^{\ \beta} f')(t). \quad (4)$$

Then the right *Riemann-Liouville* fractional derivative is defined by

$$({}^{ABR}D_b^{\alpha} f)(t) = -({}^{ABR}D_b^{\beta} f')(t), \tag{5}$$

the corresponding fractional integral is defined by

$$({}^{AB}I_b^{\alpha} f)(t) = (I_b\, {}^{AB}I_b^{\beta} f)(t). \tag{6}$$

From [5], we have

$$({}^{ABR}_{a}D^{\alpha}\, {}^{AB}_{a}I^{\alpha} f)(t) = f(t), \qquad ({}^{ABR}D_b^{\alpha}\, {}^{AB}I_b^{\alpha} f)(t) = -f(t).$$

Lemma 1. Let $f(t)$ are continuous functions on the interval $[a,b]$, $\alpha \in (1,\frac{3}{2})$, then

$$({}^{AB}_{a}I^{\alpha}\, {}^{ABC}_{a}D^{\alpha} f)(t) = f(t) - f(a) - f'(a)(t-a), \qquad ({}^{AB}I_b^{\alpha}\, {}^{ABR}D_b^{\alpha} f)(t) = f(b) - f(t),$$

$$({}^{AB}I^{\alpha}\, {}^{ABC}D_b^{\alpha} f)(t) = f(b) - f(t) - f'(b)(b-t), \qquad ({}^{AB}_{a}I^{\alpha}\, {}^{ABR}_{a}D^{\alpha} f)(t) = f(t) - f(a).$$

Proof. Using Definition 1. Let $\alpha = \beta + 1, \beta \in (0,\frac{1}{2})$, we get

$$({}^{AB}_{a}I^{\alpha}\, {}^{ABR}_{a}D^{\alpha} f)(t) = ({}_{a}I\, {}^{AB}_{a}I^{\beta}\, {}^{ABR}_{a}D^{\beta} f')(t) = {}_{a}If'(t) = f(t) - f(a).$$
$$({}^{AB}I_b^{\alpha}\, {}^{ABR}D_b^{\alpha} f)(t) = -(I_b\, {}^{AB}I_b^{\beta}\, {}^{ABR}D_b^{\beta} f')(t) = -(I_b f')(t) = f(b) - f(t).$$

$$({}^{AB}_{a}I^{\alpha}\, {}^{ABC}_{a}D^{\alpha} f)(t) = ({}_{a}I\, {}^{AB}_{a}I^{\beta}\, {}^{ABC}_{a}D^{\beta} f')(t) = {}_{a}I[f'(t) - f'(a)] = f(t) - f(a) - f'(a)(t-a).$$
$$({}^{AB}I_b^{\alpha}\, {}^{ABC}D_b^{\alpha} f)(t) = -(I_b\, {}^{AB}I_b^{\beta}\, {}^{ABC}D_b^{\beta} f')(t) = -I_b[f'(t) - f'(b)] = f(b) - f(t) - f'(b)(b-t).$$

Definition 4. ([3]) Let $f \in H^1(a,b),\ a<b,\ \alpha \in (1,\frac{3}{2})$, the left *Caputo* fractional derivative is defined by

$${}^{ABC}_{a}D^{\alpha} f(t) = \frac{1}{\Gamma(2-\alpha)} \int_a^t (t-x)^{1-\alpha} f''(x)dx,$$

the left *Riemann-Liouville* fractional derivative is defined by

$${}^{ABR}_{a}D^{\alpha} f(t) = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dt^2} \int_a^t (t-x)^{1-\alpha} f(x)dx.$$

In addition, the right *Caputo* fractional derivative is defined by

$${}^{ABC}D_b^{\alpha} f(t) = -\frac{1}{\Gamma(2-\alpha)} \int_t^b (x-t)^{1-\alpha} f''(x)dx,$$

the right *Riemann-Liouville* fractional derivative is defined by

$${}^{ABR}D_b^{\alpha} f(t) = -\frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dt^2} \int_t^b (x-t)^{1-\alpha} f(x)dx,$$

where $B(\alpha) > 0,\ B(0) = B(1) = 1$.

III. MAIN RESULTS

From[1], recall the left generalized fractional integral operator as

$$E^{\rho}_{\alpha,\beta,w,a^+}\varphi(x)=\int_a^x (x-t)^{\beta-1}E^{\rho}_{\alpha,\beta}(w(x-t)^{\alpha})\varphi(t)dt, x>a\,. \tag{7}$$

Analogously, the right generalized fractional integral operator can be defined by

$$E^{\rho}_{\alpha,\beta,w,b^-}\varphi(x)=\int_x^b (t-x)^{\beta-1}E^{\rho}_{\alpha,\beta}(w(t-x)^{\alpha})\varphi(t)dt, x<b\,. \tag{8}$$

Lemma 2. Let $f\in H^1(a,b),\ a<b,\ \alpha\in(1,\frac{3}{2})$, using (1) and (2), then the *Caputo* fractional derivative with *Mittag-Leffler* kernels is defined by

$$ {}^{ABC}_{\ a}D^{\alpha} f(t)=\frac{B(\alpha-1)}{2-\alpha}E^1_{\alpha,1,-\frac{\alpha-1}{2-\alpha},a^+} f''(t),$$

$$ {}^{ABC}D_b^{\ \alpha} f(t)=-\frac{B(\alpha-1)}{2-\alpha}E^1_{\alpha,1,-\frac{\alpha-1}{2-\alpha},b^-} f''(t)\,.$$

Then the *Riemann-Liouville* fractional derivative with *Mittag-Leffler* kernels is defined by

$$ {}^{ABR}_{\ a}D^{\alpha} f(t)=\frac{B(\alpha-1)}{2-\alpha}\frac{d^2}{dt^2}E^1_{\alpha,1,-\frac{\alpha-1}{2-\alpha},a^+} f(t),$$

$$ {}^{ABR}D_b^{\ \alpha} f(t)=-\frac{B(\alpha-1)}{2-\alpha}\frac{d^2}{dt^2}E^1_{\alpha,1,-\frac{\alpha-1}{2-\alpha},b^-} f(t)\,.$$

Proof. Because $(t-x)^{-(\alpha-1)}=\exp[-\frac{\alpha-1}{2-\alpha}(t-x)]$, and $\frac{1}{\Gamma(2-\alpha)}=\frac{B(\alpha-1)}{2-\alpha}$,

so $${}^{ABC}_{\ a}D^{\alpha} f(t)=\frac{B(\alpha-1)}{2-\alpha}\int_a^t f''(x)\exp[-\frac{\alpha-1}{2-\alpha}(t-x)]dx,$$

due to $\exp[-\frac{\alpha-1}{2-\alpha}(t-x)]=\sum_{k=0}^{\infty}\frac{[-\frac{\alpha-1}{2-\alpha}(t-x)]^k}{k!}$, so

$$\begin{aligned}{}^{ABC}_{\ a}D^{\alpha} f(t)&=\frac{B(\alpha-1)}{2-\alpha}\sum_{k=0}^{\infty}\frac{(-\frac{\alpha-1}{2-\alpha})^k}{k!}\int_a^t f''(x)(t-x)^k dx\\ &=\frac{B(\alpha-1)}{2-\alpha}\sum_{k=0}^{\infty}\frac{(-\frac{\alpha-1}{2-\alpha})^k}{\Gamma(\alpha k+1)}\int_a^t f''(x)(t-x)^{\alpha k} dx\\ &=\frac{B(\alpha-1)}{2-\alpha}E^1_{\alpha,1,\beta,a^+} f''(x).\end{aligned}$$

where $\alpha>0,\ p\geq1,\ q\geq1,\ \frac{1}{p}+\frac{1}{q}\leq1+\alpha$ ($p\neq1$ and $q\neq1$ in case $\frac{1}{p}+\frac{1}{q}=1+\alpha$), If $\varphi(x)\in L_p(a,b),\ \psi(x)\in L_q(a,b)$, then $\int_a^b \varphi(t)E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},a^+}\psi(t)dt=\int_a^b \psi(t)E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}\varphi(t)dt$.

Lemma 3. Let $f,g\in H^1(a,b),\ \alpha\in(1,\frac{3}{2}),\ \chi=-\frac{\alpha-1}{2-\alpha}$, then

$$\int_a^b g(t)\,{}^{ABC}_{\ a}D^{\alpha} f(t)dt=\int_a^b f(t)\,{}^{ABR}D_b^{\alpha} g(t)dt+\frac{B(\alpha-1)}{2-\alpha}f(t)E^1_{\alpha,1,\chi,b^-}g(t)\Big|_a^b,$$

$$\int_a^b g(t)\,{}^{ABC}D_b^\alpha f(t)dt = \int_a^b f(t)\,{}^{ABR}_{\ a}D^\alpha g(t)dt - \frac{B(\alpha-1)}{2-\alpha} f(t)E^1_{\alpha,1,\chi,a^+} g(t)\Big|_a^b .$$

Proof. Using Lemma 2, we get

$$\int_a^b g(t)\,{}^{ABC}_{\ a}D^\alpha f(t)dt = \frac{B(\alpha-1)}{2-\alpha}\int_a^b g(t)E^1_{\alpha,1,\chi,a^+} f''(t)dt$$
$$= \frac{B(\alpha-1)}{2-\alpha}\int_a^b f''(t)E^1_{\alpha,1,\chi,b^-} g(t)dt$$

$$= \frac{B(\alpha-1)}{2-\alpha} f(t)E^1_{\alpha,1,\chi,b^-} g(t)\Big|_a^b - \frac{B(\alpha-1)}{2-\alpha}\int_a^b f(t)\frac{d^2}{dt^2}E^1_{\alpha,1,\chi,b^-}dt$$

$$= \int_a^b f(t)\,{}^{ABR}D_b^\alpha g(t)dt + \frac{B(\alpha-1)}{2-\alpha} f(t)E^1_{\alpha,1,\chi,b^-} g(t)\Big|_a^b .$$

Similarly, the proof of the second part is as follows:

$$\int_a^b g(t)\,{}^{ABC}D_b^\alpha f(t)dt = \int_a^b f(t)\,{}^{ABR}_{\ a}D^\alpha g(t)dt - \frac{B(\alpha-1)}{2-\alpha} f(t)E^1_{\alpha,1,\chi,a^+} g(t)\Big|_a^b .$$

Consider the fractional *Sturm-liouville* problem with *Caputo* operator as

$${}^{ABC}_{\ a}D^\alpha (p(t)\,{}^{ABC}D_b^\alpha x(t)) + q(t)x(t) = \lambda r(t)x(t), t\in[a,b], \tag{9}$$
$${}^cL_1x(t) = {}^{ABC}_{\ a}D^\alpha (p(t)\,{}^{ABC}D_b^\alpha x(t)) + q(t)x(t),$$

where $\alpha\in(1,\frac{3}{2})$, $p(t)\neq 0$, $r(t)>0$, p,q,r are real valued continuous functions on the interval $[a,b]$, and boundary conditions

$$c_1E^1_{\alpha,1,\chi,b^-}x(a) + c_2\,{}^{ABC}D_b^\alpha x(a) = 0, \tag{10}$$
$$d_1E^1_{\alpha,1,\chi,b^-}x(b) + d_2\,{}^{ABC}D_b^\alpha x(b) = 0, \tag{11}$$

where $c_1^2+c_2^2\neq 0$, $d_1^2+d_2^2\neq 0$.

Theorem 1. The fractional *Caputo* operator ${}^cL_1x(t)$ is self-adjoint.

Proof. We have

$$\varphi(t)\,{}^cL_1\overline{\psi(t)} = \varphi(t)\,{}^{ABC}_{\ a}D^\alpha (p(t)\,{}^{ABC}D_b^\alpha \overline{\psi(t)}) + q(t)\overline{\psi(t)}\varphi(t), \tag{12}$$
$$\overline{\psi(t)}\,{}^cL_1\varphi(t) = \overline{\psi(t)}\,{}^{ABC}_{\ a}D^\alpha (p(t)\,{}^{ABC}D_b^\alpha \varphi(t)) + q(t)\overline{\psi(t)}\varphi(t). \tag{13}$$

Let (12) subtract to (13), we have

$$\varphi(t)\,{}^cL_1\overline{\psi(t)} - \overline{\psi(t)}\,{}^cL_1\varphi(t) = \varphi(t)\,{}^{ABC}_{\ a}D^\alpha (p(t)\,{}^{ABC}D_b^\alpha \overline{\psi(t)}) - \overline{\psi(t)}\,{}^{ABC}_{\ a}D^\alpha (p(t)\,{}^{ABC}D_b^\alpha \varphi(t)).$$

Integrating from *a* to *b*, we have

$$\int_a^b [\varphi(t)\,{}^cL_1\overline{\psi(t)} - \overline{\psi(t)}\,{}^cL_1\varphi(t)]dt$$
$$= \int_a^b [\varphi(t)\,{}^{ABC}_{\ a}D^\alpha (p(t)\,{}^{ABC}D_b^\alpha \overline{\psi(t)})]dt - \int_a^b [\overline{\psi(t)}\,{}^{ABC}_{\ a}D^\alpha (p(t)\,{}^{ABC}D_b^\alpha \varphi(t))]dt$$
$$= \int_a^b [p(t)\,{}^{ABC}D_b^\alpha \overline{\psi(t)}\,{}^{ABC}D_b^\alpha \varphi(t)]dt - \int_a^b [p(t)\,{}^{ABC}D_b^\alpha \varphi(t)\,{}^{ABC}D_b^\alpha \overline{\psi(t)}]dt$$
$$= 0.$$

Hence, $<{}^{c}L_1\varphi,\psi>=<\varphi,{}^{c}L_1\psi>$. That is, ${}^{c}L_1$ is self-adjoint.
Consider the fractional *Sturm-liouville* problem with *Riemann-Liouville* operator as

$$ {}^{ABR}_{a}D^{\alpha}(p(t)\,{}^{ABR}D^{\alpha}_{b}x(t))+q(t)x(t)=\lambda r(t)x(t), t\in[a,b], \tag{14}$$

$$ {}^{R}L_1x(t)={}^{ABR}_{a}D^{\alpha}(p(t)\,{}^{ABR}D^{\alpha}_{b}x(t))+q(t)x(t).$$

where $\alpha\in(1,\frac{3}{2}),\ p(t)\neq 0,\ r(t)>0,\ p,q,r$ are real valued continuous functions on the interval $[a,b]$, and boundary conditions

$$ c_1E^{1}_{\alpha,1,\chi,b^-}x(a)+c_2\,{}^{ABR}D^{\alpha}_{b}x(a)=0, \tag{15}$$

$$ d_1E^{1}_{\alpha,1,\chi,b^-}x(b)+d_2\,{}^{ABR}D^{\alpha}_{b}x(b)=0. \tag{16}$$

where $c_1^2+c_2^2\neq 0,\ d_1^2+d_2^2\neq 0$.

Theorem 2. The fractional *Riemann-Liouville* operator ${}^{R}L_1x(t)$ is self-adjoint.
Proof. The proof is similar to that of Theorem 1.

Theorem 3. The eigenvalues of the *Sturm-Liouville* equations with *Caputo* operator are real.
Proof. Assume that λ is the eigenvalue for (9) corresponding to eigenfunction x, then x and its complex conjugate $\bar{x}$ satisfy

$$ {}^{c}L_1x(t)=\lambda r(t)x(t), \tag{17}$$

$$ {}^{c}L_1\bar{x}(t)=\bar{\lambda}r(t)\bar{x}(t). \tag{18}$$

We multiply (17) by $\bar{x}(t)$ and (18) by $x(t)$, respectively, and subtract to obtain

$$ (\bar{\lambda}-\lambda)r(t)x(t)\bar{x}(t)=x(t)\,{}^{c}L_1\bar{x}(t)-\bar{x}(t)\,{}^{c}L_1x(t).$$

Integrating from *a* to *b*, and using Theorem 1, we get

$$ (\bar{\lambda}-\lambda)\int_a^b r(t)|x(t)|^2dt=0,$$

and $r(t)>0$, we conclude that $\bar{\lambda}=\lambda$.

Theorem 4. The eigenvalues of the *Sturm-Liouville* equations with *Caputo* operator are real.
Proof. The proof is similar to that of Theorem 3.

Theorem 5. The eigenfunctions, corresponding to distinct eigenvalues of the *Sturm-Liouville* equations (9) are orthogonal with respect to the weight function r on $[a,b]$ that is

$$ \langle x_{\lambda_1},x_{\lambda_2}\rangle=\int_a^b r(t)x_{\lambda_1}(t)x_{\lambda_2}(t)dt=0,\quad \lambda_1\neq\lambda_2,$$

When the functions x_{λ_i} correspond to eigenvalues $\lambda_i,\ i=1,2$.

Proof. Let λ_1 and λ_2 be two distinct eigenvalues of (9) corresponding to the eigenfuntions x_{λ_1} and x_{λ_2}, respectively. Then we have

$$ {}^{ABC}_{a}D^{\alpha}(p(t)\,{}^{ABC}D^{\alpha}_{b}x_{\lambda_1}(t))+q(t)x_{\lambda_1}(t)=\lambda_1r(t)x_{\lambda_1}(t), \tag{19}$$

$$ {}^{ABC}_{a}D^{\alpha}(p(t)\,{}^{ABC}D^{\alpha}_{b}x_{\lambda_2}(t))+q(t)x_{\lambda_2}(t)=\lambda_2r(t)x_{\lambda_2}(t). \tag{20}$$

We multiply (19) and (20) by $x_{\lambda_2}(t)$ and $x_{\lambda_1}(t)$, respectively, and subtract the results to obtain

$$x_{\lambda_2}\, {}^{ABC}_{a}D^{\alpha}(p(t)\, {}^{ABC}D^{\alpha}_{b} x_{\lambda_1}(t)) - x_{\lambda_1}(t)\, {}^{ABC}_{a}D^{\alpha}(p(t)\, {}^{ABC}D^{\alpha}_{b} x_{\lambda_2}(t)) = (\lambda_1 - \lambda_2) r(t) x_{\lambda_1}(t) x_{\lambda_2}(t).$$

Now, integrating from *a* to *b*, we get

$$(\lambda_1 - \lambda_2)\int_a^b r(t) x_{\lambda_1}(t) x_{\lambda_2}(t) = 0 .$$

Since $\lambda_1 \neq \lambda_2$, it follows that

$$\int_a^b r(t) x_{\lambda_1}(t) x_{\lambda_2}(t) = 0 ,$$

which completes the proof.

Theorem 6. The eigenfunctions, corresponding to distinct eigenvalues of the *Sturm-Liouville* equations (14) are orthogonal with respect to the weight function r on $[a,b]$ that is

$$\langle x_{\lambda_1}, x_{\lambda_2} \rangle = \int_a^b r(t) x_{\lambda_1}(t) x_{\lambda_2}(t) dt = 0, \quad \lambda_1 \neq \lambda_2 ,$$

When the functions x_{λ_i} correspond to eigenvalues λ_i, $i = 1,2$.

Proof. The proof is similar to that of Theorem 5.

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