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Existence of positive solutions for fractional *q*-difference equations involving integral boundary conditions with *p*-Laplacian operator

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ABSTRACT: The existence of positive solutions is considered for a fractional q-difference equation with p-Laplacian operator in this article. By employing the Avery-Henderson fixed point theorem, a new result is obtained for the boundary value problems.

KEY WORDS: positive solutions; Riemann-Liouville fractional q-derivatives; Caputo fractional q-derivatives; p-Laplacian; Avery-Henderson fixed point theorem

I. INTRODUCTION

Fractional calculus is the extension of integer order calculus to arbitrary order calculus. With the development of fractional calculus, fractional differential equations have wide applications in the modeling of different physical and natural science fields, such as fluid mechanics, chemistry, control system, heat conduction, etc. There are many papers concerning fractional differential equations with the p-Laplacian operator [1-5] and fractional differential equations with integral boundary conditions [6-10].

In [11], Yang studied the following fractional q-difference boundary value problem with p-Laplacian operator

$$
\begin{aligned}
&{}^cD_q^\beta(\phi_p({}^cD_q^\alpha x(t))) = f(t,u(t)), 0<t<1, 2<\alpha<3,\\
&x(0)=x(1)=0, \qquad {}^cD_q^\alpha x(1) = {}^cD_q^\alpha x(0) = 0,
\end{aligned}
$$

where $1<\alpha,\beta\le 2$. The existence results for the above boundary value problem were obtained by using the upper and lower solutions method associates with the Schauder fixed point theorem.

In [12], Yuan and Yang considered a class of four-point boundary value problems of fractional q-difference equations with p-Laplacian operator

$$
\begin{aligned}
&D_q^\beta(\phi_P(D_q^\alpha x(t))) = f(t,u(t)), 0<t<1, 2<\alpha<3,\\
&x(0)=0, x(1)=ax\xi, \qquad D_q^\alpha x(0)=0, D_q^\alpha x(1) = bD_q^\alpha x(\eta),
\end{aligned}
$$

Where D_q^β, D_q^α is the fractional q-derivative of the Riemann-Liouville type with $1<\alpha,\beta\le 2$. By applying the upper and lower solutions method associated with the Schauder fixed point theorem, the existence results of at least one positive solution for the above fractional q-difference boundary value problem with p-Laplacian operator are established. Motivated by the aforementioned work, this work discusses the existence of positive solutions for this fractional q-difference equation:

$$\begin{cases} D^{\beta}_{q,0^+}[\phi_p({}^cD^{\alpha}_{q,0^+}u(t))]+f(t,u(t))=0, t\in(0,1), \\ \phi_p({}^cD^{\alpha}_{q,0^+}u(0))=[\phi_p({}^cD^{\alpha}_{q,0^+}u(0))]'=\phi_p({}^cD^{\alpha}_{q,0^+}u(1))=0, \\ D^2_{q,0^+}u(0)=D_{q,0^+}u(1)=0, \\ au(0)+bD_{q,0^+}u(0)=\int_0^1 g(t)u(t)d_qt, \end{cases} \tag{1.1}$$

where $2<\alpha\le 3, 2<\beta\le 3,$ and $5<\alpha+\beta\le 6$. $\phi_p(u)=|u|^{p-2}u,\ p>1$. ${}^cD^{\alpha}_{q,0^+}$ is the Caputo fractional *q*-derivative, $D^{\beta}_{q,0^+}$ is the Riemann-Liouville fractional *q*-derivative.

We will always suppose the following conditions are satisfied:

(H_1) $g(t):[0,1]\to[0,+\infty)$ with $g(t)\in L^1[0,1],$ $\int_0^1 g(t)d_qt>0,$ and $\int_0^1 tg(t)d_qt>0;$

(H_2) $a,b\in(0,+\infty),$ $a>\int_0^1 g(t)d_qt$ and $b>a;$

(H_3) $f(t,u):[0,1]\times(0,+\infty)\to(0,+\infty)$ is continuous.

II. BACKGROUND AND DEFINITIONS

To show the main result of this work, we give in the following some basic definitions and theorems, which can be found in [13,14].

Definition 2.1 [13] *Let* $\alpha\ge 0$ *and* f *be a function defined on* $[0,1]$. *The fractional q-integral of Riemann-Liouville type is* $(I^0_{q,0^+}f)(t)$ *and* $f(t)$

$$(I^{\alpha}_{q,0^+}f)(t)=\int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\nu)}f(s)d_qs, \qquad \alpha>0, t\in[0,1].$$

Lemma 2.2 [13] *Let* $\alpha>0,$ *then the following equality holds:*

$$I^{\alpha}_{q,0^+}\,{}^cD^{\alpha}_{q,0^+}f(t)=f(t)-\sum_{k=0}^{[\alpha]-1}\frac{t^k}{\Gamma_q(k+1)}(D^k_{q,0^+}f)(0^+).$$

Lemma 2.3 [14]*Let* $\alpha>0,$ *If* $D^{\alpha}_{q,0^+}u\in C[0,1],$ *then*

$$I^{\alpha}_{q,0^+}D^{\alpha}_{q,0^+}f(t)=f(t)+\sum_{i=1}^{n}k_it^{(\alpha-i)}$$

where $n=[\alpha]+1.$

$$\omega(0)=0,$$

Theorem 2.4 (Avery-Henderson fixed point theorem [15]) *Let* $(E,\|\cdot\|)$ *be a Banach space, and* $P\subset E$ *be a cone. Let* ψ *and* φ *be increasing non-negative, continuous functionals on* P, *and* ω *be a non-negative continuous functionals on* P *with such that, for some* $r_3>0$ *and* $M>0,$ $\varphi(u)<\omega(u)<\psi(u),$

and $\|u\| \le M\varphi(u)$, *for all* $u \in \overline{p(\varphi, r_3)}$, *where* $p(\varphi, r_3) = \{u \in P : \varphi(u) < r_3\}$. *Suppose that there exist positive numbers* $r_1 < r_2 < r_3$, *such that*

$$\omega(lu) \le l\omega(u) \quad \text{for } 0 \le l \le 1, \text{ and } u \in \partial P(\omega, r_2).$$

If $T : \overline{p(\varphi, r_3)} \to P$ *is a completely continuous operator satisfying:*

(C_1) $\varphi(Tu) > r_3$ *for all* $u \in \partial P(\omega, r_3)$;

(C_2) $\varphi(Tu) < r_2$ *for all* $u \in \partial P(\omega, r_2)$;

(C_3) $P(\psi, r_1) \ne \phi$, *and* $\psi(Tu) > r_1$ *for all* $u \in \partial P(\psi, r_1)$,

then T *has at least two fixed points* u_1 *and* u_2 *such that* $r_1 < \psi(u_1)$ *with* $\omega(u_1) < r_2$ *and* $r_2 < \omega(u_2)$ *with* $\varphi(u_2) < r_3$.

III. PRELIMINARY LEMMAS

Lemma 3.1 *The boundary value problem* (1.1) *is equivalent to the following equation:*

$$u(t) = d_0 + d_1 t + \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} v(s) d_q s, \tag{3.1}$$

where

$$\begin{aligned} d_0 = {} & \frac{1}{[a - \int_0^1 g(t) d_q t]\Gamma_q(\alpha)} \int_0^1 g(t) \int_0^t (t-qs)^{(\alpha-1)} v(s) d_q s d_q t \\ & + \frac{[b - \int_0^1 t g(t) d_q t]}{[a - \int_0^1 g(t) d_q t]\Gamma_q(\alpha-1)} \int_0^1 (1-qs)^{(\alpha-2)} v(s) d_q s, \end{aligned} \tag{3.2}$$

$$d_1 = -\frac{1}{\Gamma_q(\alpha-1)} \int_0^1 (1-qs)^{(\alpha-2)} v(s) d_q s, \tag{3.3}$$

$$v(s) = \phi_q\left(\int_0^1 H(s, q\tau) f(\tau, u(\tau)) d_q \tau\right), \tag{3.4}$$

$$H(s,\tau) = \frac{1}{\Gamma_q(\beta)} \begin{cases} (s(1-\tau))^{(\beta-1)} - (s-\tau)^{(\beta-1)}, & 0 \le \tau \le s \le 1, \\ (s(1-\tau))^{(\beta-1)}, & 0 \le s \le \tau \le 1. \end{cases} \tag{3.5}$$

$\phi_q(s)$ *is the inverse function of* $\phi_p(s)$, *a.e.*, $\phi_q(s) = |s|^{q-2} s$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof From $D^{\beta}_{q,0^+}[\phi_p({}^c D^{\alpha}_{q,0^+} u(t))] + f(t, u(t)) = 0$, we get

$$\phi_p({}^c D^{\alpha}_{q,0^+} u(t)) = -\frac{1}{\Gamma_q(\beta)} \int_0^t (t-q\tau)^{(\beta-1)} f(\tau, u(\tau)) d_q \tau + c_1 t^{\beta-1} + c_2 t^{\beta-2} + c_3 t^{\beta-3}.$$

In view of $\phi_p({}^cD^{\alpha}_{q,0^+}u(0)) = [\phi_p({}^cD^{\alpha}_{q,0^+}u(0))]' = 0,$ we get $c_2 = c_3 = 0,$ i.e.,

$$\phi_p({}^cD^{\alpha}_{q,0^+}u(t)) = -\frac{1}{\Gamma_q(\beta)}\int_0^t (t-q\tau)^{(\beta-1)} f(\tau,u(\tau))d_q\tau + c_1 t^{\beta-1}. \tag{3.6}$$

Conditions $\phi_p({}^cD^{\alpha}_{q,0^+}u(1)) = 0$ imply that

$$c_1 = -\frac{1}{\Gamma_q(\beta)}\int_0^1 (1-q\tau)^{(\beta-1)} f(\tau,u(\tau))d_q\tau. \tag{3.7}$$

By use of (3.6) and (3.7), we get

$$\phi_p({}^cD^{\alpha}_{q,0^+}u(t)) = \int_0^1 H(t,q\tau) f(\tau,u(\tau))d_q\tau. \tag{3.8}$$

In view of (3.8), we obtain

$${}^cD^{\alpha}_{q,0^+}u(t) = \phi_q\left(\int_0^1 H(t,q\tau) f(\tau,u(\tau))d_q\tau\right). \tag{3.9}$$

Let

$$v(t) = \phi_q\left(\int_0^1 H(t,q\tau) f(\tau,u(\tau))d_q\tau\right),$$

by use of (3.9), we get

$$u(t) = \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} v(s)d_qs + d_0 + d_1 t + d_2 t^2.$$

Conditions $D^2_{q,0^+}u(0) = 0$ imply that $d_2 = 0,$ i.e.,

$$u(t) = \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} v(s)d_qs + d_0 + d_1 t,$$

then we have

$$u'(t) = \frac{1}{\Gamma_q(\alpha-1)}\int_0^t (t-qs)^{(\alpha-2)} v(s)d_qs + d_1.$$

Conditions $D_{q,0^+}u(1) = 0$ imply that

$$d_1 = -\frac{1}{\Gamma_q(\alpha-1)}\int_0^1 (1-qs)^{(\alpha-2)} v(s)d_qs.$$

From $au(0) + bD_{q,0^+}u(0) = \int_0^1 g(t)u(t)d_qt,$ we get

$$d_0 = \frac{1}{[a-\int_0^1 g(t)d_qt]\Gamma_q(\alpha)}\int_0^1 g(t)\int_0^t (t-qs)^{(\alpha-1)} v(s)d_qs d_qt$$
$$+\frac{[b-\int_0^1 tg(t)d_qt]}{[a-\int_0^1 g(t)d_qt]\Gamma_q(\alpha-1)}\int_0^1 (1-qs)^{(\alpha-2)} v(s)d_qs.$$

Therefore, we can obtain

$$u(t) = d_0 + d_1 t + \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} v(s)d_qs$$

$$= \frac{1}{[a-\int_0^1 g(t)d_qt]\Gamma_q(\alpha)}\int_0^1 g(t)\int_0^t (t-qs)^{(\alpha-1)}v(s)d_qsd_qt$$

$$+\frac{[b-\int_0^1 tg(t)d_qt]}{[a-\int_0^1 g(t)d_qt]\Gamma_q(\alpha-1)}\int_0^1 (1-qs)^{(\alpha-2)}v(s)d_qs$$

$$-\frac{t}{\Gamma_q(\alpha-1)}\int_0^1 (1-qs)^{(\alpha-2)}v(s)d_qs+\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}v(s)d_qs.$$

The proof is complete. □

Lemma 3.2 ([16]) The *function* $H(s,\tau)$ *defined by* (3.5) *is continuous on* $[0,1]\times[0,1]$ *and satisfy*

$$\frac{s^{\beta-1}(1-s)\tau(1-\tau)^{(\beta-1)}}{\Gamma_q(\beta)}\le H(s,\tau)\le\frac{\tau(1-\tau)^{(\beta-1)}}{\Gamma_q(\beta-1)} \quad \textit{for} \quad s,\tau\in[0,1].$$

Let E be the real Banach space $C[0,1]$ with the maximum norm, define the operator $T:E\to E$ by

$$Tu(t)=d_0+d_1t+\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}v(s)d_qs$$

$$= \frac{1}{[a-\int_0^1 g(t)d_qt]\Gamma_q(\alpha)}\int_0^1 g(t)\int_0^t (t-qs)^{(\alpha-1)}v(s)d_qsd_qt$$

$$+\frac{[b-\int_0^1 tg(t)d_qt]}{[a-\int_0^1 g(t)d_qt]\Gamma_q(\alpha-1)}\int_0^1 (1-qs)^{(\alpha-2)}v(s)d_qs$$

$$-\frac{t}{\Gamma_q(\alpha-1)}\int_0^1 (1-qs)^{(\alpha-2)}v(s)d_qs+\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}v(s)d_qs.$$

Lemma 3.3 *For* $u\in C[0,1]$ *with* $u(t)\ge 0,$ $(Tu)(t)$ *is non-increasing and non-negative.*

Proof Since

$$Tu(t)=d_0+d_1t+\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}v(s)d_qs,$$

so we get

$$(Tu)'(t)=d_1+\frac{1}{\Gamma_q(\alpha-1)}\int_0^t (t-qs)^{(\alpha-2)}v(s)d_qs$$

$$=-\frac{1}{\Gamma_q(\alpha-1)}\int_0^1 (1-qs)^{(\alpha-2)}v(s)d_qs+\frac{1}{\Gamma_q(\alpha-1)}\int_0^t (t-qs)^{(\alpha-2)}v(s)d_qs$$

$$\le 0.$$

So $(Tu)(t)$ is non-increasing, then we have $\min_{t\in[0,1]} Tu(t)=Tu(1).$ We have

$$Tu(1)=d_0+d_1+\frac{1}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-1)}v(s)d_qs$$

$$
\begin{aligned}
&= \frac{1}{[a-\int_0^1 g(t)d_qt]\Gamma_q(\alpha)}\int_0^1 g(t)\int_0^t (t-qs)^{(\alpha-1)}v(s)d_qsd_qt \\
&\quad + \frac{[b-\int_0^1 tg(t)d_qt]}{[a-\int_0^1 g(t)d_qt]\Gamma_q(\alpha-1)}\int_0^1 (1-qs)^{(\alpha-2)}v(s)d_qs \\
&\quad - \frac{1}{\Gamma_q(\alpha-1)}\int_0^1 (1-qs)^{(\alpha-2)}v(s)d_qs + \frac{1}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-1)}v(s)d_qs \\
&\geq \frac{1}{[a-\int_0^1 g(t)d_qt]\Gamma_q(\alpha)}\int_0^1 g(t)\int_0^t (t-qs)^{(\alpha-1)}v(s)d_qsd_qt \\
&\quad + \frac{[b-\int_0^1 tg(t)d_qt]-[a-\int_0^1 g(t)d_qt]}{[a-\int_0^1 g(t)d_qt]\Gamma_q(\alpha-1)}\int_0^1 (1-qs)^{(\alpha-2)}v(s)d_qs \\
&\quad + \frac{1}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-1)}v(s)d_qs \\
&\geq 0.
\end{aligned}
$$

The proof is complete. □

IV. MAIN RESULTS

Theorem 4.1 Suppose *that there exist numbers* $0 \leq r_1 \leq r_2 \leq r_3$ *such that f satisfies the following conditions:*

(H_1) $f(t,u) > M_3$, *for* $t \in [0,1]$, $u \in [r_3, \frac{r_3}{k}]$;

(H_2) $f(t,u) < M_2$, *for* $t \in [0,1]$, $u \in [0, r_2]$;

(H_3) $f(t,u) > M_1$, *for* $t \in [0,1]$, $u \in [0, r_1]$,

where

$$
M_3 = \frac{\Gamma_q(\beta)}{qB_q(2,\beta)}\left(\frac{r_3}{L_3}\right)^{p-1}, \qquad M_2 = \frac{\Gamma_q(\beta-1)}{qB_q(2,\beta)}\left(\frac{r_2}{L_2}\right)^{p-1}, \qquad M_1 = \frac{\Gamma_q(\beta)}{qB_q(2,\beta)}\left(\frac{r_1}{L_1}\right)^{p-1},
$$

$$
L_3 = \left\{\frac{b-a}{[a-\int_0^1 g(t)d_qt]\Gamma_q(\alpha)} + \frac{1}{\Gamma_q(\alpha+1)}\right\}\int_0^1 \left(s^{\beta-1}(1-s)\right)^{q-1} d_qs,
$$

$$
L_2 = \frac{\int_0^1 g(t)d_qt}{[a-\int_0^1 g(t)d_qt]\Gamma_q(\alpha+1)} + \frac{b-\int_0^1 tg(t)d_qt}{[a-\int_0^1 g(t)d_qt]\Gamma_q(\alpha)},
$$

$$
L_1 = \frac{b-\int_0^1 tg(t)d_qt}{[a-\int_0^1 g(t)d_qt]\Gamma_q(\alpha)}\int_0^1 \left(s^{\beta-1}(1-s)\right)^{q-1} d_qs.
$$

Then the problem (1.1) *has at least two positive solutions* u_1 *and* u_2 *such that* $r_1 < \psi(u_1)$ *with* $\omega(u_1) < r_2$

and $r_2 < \omega(u_2)$ *with* $\varphi(u_2) < r_3$.

Proof Define the cone $P \subset E$ by

$$P = \left\{ u \middle| u \in E, \min_{t\in[0,1]} u(t) \ge k\|u\|, t \in [0,1] \right\},$$

where

$$k = \frac{[b - \int_0^1 tg(t)d_qt] - [a - \int_0^1 g(t)d_qt]}{b - \int_0^1 tg(t)d_qt}, \qquad 0 < k < 1.$$

For any $u \in P$, in view of Lemma 3.3, we get

$$\begin{aligned}
\min_{t\in[0,1]} |Tu(t)| = |Tu(1)| &= d_0 + d_1 + \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} v(s) d_qs \\
&= \frac{1}{[a - \int_0^1 g(t)d_qt]\Gamma_q(\alpha)} \int_0^1 g(t) \int_0^t (t-qs)^{(\alpha-1)} v(s) d_qs d_qt \\
&\quad + \frac{[b - \int_0^1 tg(t)d_qt]}{[a - \int_0^1 g(t)d_qt]\Gamma_q(\alpha-1)} \int_0^1 (1-qs)^{(\alpha-2)} v(s) d_qs \\
&\quad - \frac{1}{\Gamma_q(\alpha-1)} \int_0^1 (1-qs)^{(\alpha-2)} v(s) d_qs + \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} v(s) d_qs \\
&\ge \frac{1}{[a - \int_0^1 g(t)d_qt]\Gamma_q(\alpha)} \int_0^1 g(t) \int_0^t (t-qs)^{(\alpha-1)} v(s) d_qs d_qt \\
&\quad + \frac{[b - \int_0^1 tg(t)d_qt] - [a - \int_0^1 g(t)d_qt]}{[a - \int_0^1 g(t)d_qt]\Gamma_q(\alpha-1)} \int_0^1 (1-qs)^{(\alpha-2)} v(s) d_qs \\
&\ge k \left\{ \frac{1}{[a - \int_0^1 g(t)d_qt]\Gamma_q(\alpha)} \int_0^1 g(t) \int_0^t (t-qs)^{(\alpha-1)} v(s) d_qs d_qt \right. \\
&\quad \left. + \frac{[b - \int_0^1 tg(t)d_qt]}{[a - \int_0^1 g(t)d_qt]\Gamma_q(\alpha-1)} \int_0^1 (1-qs)^{(\alpha-2)} v(s) d_qs \right\} \\
&= kTu(0) = k\|Tu\|.
\end{aligned}$$

Therefore, $T: P \to P$. In view of the Arzela-Ascoli theorem, we have $T: P \to P$ is completely continuous.

We define the functions on the cone *P*:

$$\varphi(u) = \min_{t\in[0,1]} |u(t)| = u(1), \qquad \omega(u) = \max_{t\in[0,1]} |u(t)| = u(0),$$

$$\psi(u) = \max_{t\in[0,1]} |u(t)| = u(0).$$

Obviously, we have $\omega(0) = 0,\ \varphi(u) \le \omega(u) \le \psi(u)$.

For any $u \in \overline{p(\varphi, r_3)}$, we get $\min_{t\in[0,1]} u(t) \ge k\|u\|$, that is, $\varphi(u) \ge k\|u\|$, therefore we obtain $\|u\| \le \frac{1}{k}\varphi(u)$.

For any $u \in \partial P(\omega, r_2)$, we get $\omega(lu) = l\omega(u)$ for $0 \le l \le 1$.

In the following, we prove that the conditions of Theorem 2.1 hold.

Firstly, let $u \in \partial P(\omega, r_3)$, that is, $u \in [\frac{r_3}{k}, r_3]$ for $t \in [0,1]$. By means of (H_1), we have

$$
\begin{aligned}
v(s) &= \phi_q\left(\int_0^1 H(s,q\tau) f(\tau,u(\tau)) d_q\tau\right) \\
&> \phi_q\left(M_3 \int_0^1 \frac{s^{\beta-1}(1-s) q\tau(1-q\tau)^{(\beta-1)}}{\Gamma_q(\beta)} d_q\tau\right) \\
&= \left(\frac{M_3 s^{\beta-1}(1-s) q B_q(2,\beta)}{\Gamma_q(\beta)}\right)^{q-1},
\end{aligned}
$$

where $B_q(2,\beta) = \int_0^1 \tau(1-q\tau)^{(\beta-1)} d_q\tau$. So we get

$$
\begin{aligned}
\varphi(Tu) &= \min_{t\in[0,1]} |Tu(t)| = Tu(1) = d_0 + d_1 + \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} v(s) d_q s \\
&= \frac{1}{[a - \int_0^1 g(t) d_q t]\Gamma_q(\alpha)} \int_0^1 g(t) \int_0^t (t-qs)^{(\alpha-1)} v(s) d_q s d_q t \\
&\quad + \frac{[b - \int_0^1 t g(t) d_q t]}{[a - \int_0^1 g(t) d_q t]\Gamma_q(\alpha-1)} \int_0^1 (1-qs)^{(\alpha-2)} v(s) d_q s \\
&\quad - \frac{1}{\Gamma_q(\alpha-1)} \int_0^1 (1-qs)^{(\alpha-2)} v(s) d_q s + \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} v(s) d_q s \\
&\ge \frac{[b - \int_0^1 t g(t) d_q t] - [a - \int_0^1 g(t) d_q t]}{[a - \int_0^1 g(t) d_q t]\Gamma_q(\alpha-1)} \int_0^1 (1-qs)^{(\alpha-2)} v(s) d_q s \\
&\quad + \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} v(s) d_q s \\
&> \frac{b-a}{[a - \int_0^1 g(t) d_q t]\Gamma_q(\alpha-1)} \int_0^1 (1-qs)^{(\alpha-2)} \left(\frac{M_3 s^{\beta-1}(1-s) q B_q(2,\beta)}{\Gamma_q(\beta)}\right)^{q-1} d_q s \\
&\quad + \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} \left(\frac{M_3 s^{\beta-1}(1-s) q B_q(2,\beta)}{\Gamma_q(\beta)}\right)^{q-1} d_q s \\
&= \frac{b-a}{[a - \int_0^1 g(t) d_q t]\Gamma_q(\alpha)} \left(\frac{M_3 q B_q(2,\beta)}{\Gamma_q(\beta)}\right)^{q-1} \int_0^1 \left(s^{\beta-1}(1-s)\right)^{q-1} d_q s \\
&\quad + \frac{1}{\Gamma_q(\alpha+1)} \left(\frac{M_3 q B_q(2,\beta)}{\Gamma_q(\beta)}\right)^{q-1} \int_0^1 \left(s^{\beta-1}(1-s)\right)^{q-1} d_q s \\
&= \left(\frac{M_3 q B_q(2,\beta)}{\Gamma_q(\beta)}\right)^{q-1} L_3 = r_3.
\end{aligned}
$$

Secondly, let $u \in \partial P(\omega, r_2)$, that is, $u \in [0, r_2]$ for $t \in [0,1]$. By means of (H_2), we get

$$v(s) = \phi_q\left(\int_0^1 H(s,q\tau) f(\tau,u(\tau)) d_q\tau\right)$$
$$< \phi_q\left(M_2\int_0^1 \frac{q\tau(1-q\tau)^{(\beta-1)}}{\Gamma_q(\beta-1)} d_q\tau\right) = \left(\frac{M_2 q B_q(2,\beta)}{\Gamma_q(\beta-1)}\right)^{q-1}.$$

So, we have

$$\omega(Tu) = \max_{t\in[0,1]} |Tu(t)| = Tu(0) = d_0$$
$$= \frac{1}{[a-\int_0^1 g(t)d_q t]\Gamma_q(\alpha)} \int_0^1 g(t)\int_0^t (t-qs)^{(\alpha-1)} v(s) d_q s d_q t$$
$$+ \frac{[b-\int_0^1 tg(t)d_q t]}{[a-\int_0^1 g(t)d_q t]\Gamma_q(\alpha-1)} \int_0^1 (1-qs)^{(\alpha-2)} v(s) d_q s$$
$$< \frac{1}{[a-\int_0^1 g(t)d_q t]\Gamma_q(\alpha)} \int_0^1 g(t)\int_0^t (t-qs)^{(\alpha-1)} \left(\frac{M_2 q B_q(2,\beta)}{\Gamma_q(\beta-1)}\right)^{q-1} d_q s d_q t$$
$$+ \frac{[b-\int_0^1 tg(t)d_q t]}{[a-\int_0^1 g(t)d_q t]\Gamma_q(\alpha-1)} \int_0^1 (1-qs)^{(\alpha-2)} \left(\frac{M_2 q B_q(2,\beta)}{\Gamma_q(\beta-1)}\right)^{q-1} d_q s$$
$$< \frac{\int_0^1 g(t)d_q t}{[a-\int_0^1 g(t)d_q t]\Gamma_q(\alpha+1)} \left(\frac{M_2 q B_q(2,\beta)}{\Gamma_q(\beta-1)}\right)^{q-1}$$
$$+ \frac{[b-\int_0^1 tg(t)d_q t]}{[a-\int_0^1 g(t)d_q t]\Gamma_q(\alpha)} \left(\frac{M_2 q B_q(2,\beta)}{\Gamma_q(\beta-1)}\right)^{q-1}$$
$$= \left(\frac{M_2 q B_q(2,\beta)}{\Gamma_q(\beta-1)}\right)^{q-1} L_2 = r_2.$$

Finally, let $u \in \partial P(\psi, r_1)$, that is, $u \in [0, r_1]$ for $t \in [0,1]$. By means of (H_3), we get

$$v(s) = \phi_q\left(\int_0^1 H(s,q\tau) f(\tau,u(\tau)) d_q\tau\right)$$
$$> \phi_q\left(M_1\int_0^1 \frac{s^{\beta-1}(1-s)q\tau(1-q\tau)^{(\beta-1)}}{\Gamma_q(\beta)} d_q\tau\right) = \left(\frac{M_1 s^{\beta-1}(1-s) q B_q(2,\beta)}{\Gamma_q(\beta)}\right)^{q-1}.$$

So we get

$$\psi(Tu) = \max_{t\in[0,1]} |Tu(t)| = Tu(0) = d_0$$
$$= \frac{1}{[a-\int_0^1 g(t)d_q t]\Gamma_q(\alpha)} \int_0^1 g(t)\int_0^t (t-qs)^{(\alpha-1)} v(s) d_q s d_q t$$

$$\geq \frac{[b-\int_0^1 tg(t)d_qt]}{[a-\int_0^1 g(t)d_qt]\Gamma_q(\alpha-1)}\int_0^1 (1-qs)^{(\alpha-2)}v(s)d_qs$$

$$> \frac{[b-\int_0^1 tg(t)d_qt]}{[a-\int_0^1 g(t)d_qt]\Gamma_q(\alpha-1)}\int_0^1 (1-qs)^{(\alpha-2)}\left(\frac{M_1 s^{\beta-1}(1-s)qB_q(2,\beta)}{\Gamma_q(\beta)}\right)^{q-1} d_qs$$

$$= \frac{[b-\int_0^1 tg(t)d_qt]}{[a-\int_0^1 g(t)d_qt]\Gamma_q(\alpha)}\left(\frac{M_1 qB_q(2,\beta)}{\Gamma_q(\beta)}\right)^{q-1}\int_0^1 \left(s^{\beta-1}(1-s)\right)^{q-1} d_qs$$

$$= \left(\frac{M_1 qB_q(2,\beta)}{\Gamma_q(\beta)}\right)^{q-1} L_1 = r_1.$$

Therefore, in view of Theorem 2.1, we see that the problem (1.1) has at least two positive solutions u_1 and u_2 such that $r_1 < \psi(u_1)$ with $\omega(u_1) < r_2$ and $r_2 < \omega(u_2)$ with $\varphi(u_2) < r_3$. □

VI. EXAMPLE

In this section, we give a simple example to explain the main result.

Example 5.1 *For the problem* (1.1)*, let* $\alpha = 2.8,\ \beta = 2.3,\ a = 4,\ b = 10,\ p = 2,\ g(t) = t,$ *then we get* $q = 2,$ $\int_0^1 g(t)d_qt = \frac{1}{2},\quad \int_0^1 tg(t)d_qt = \frac{1}{3},$

$$k = \frac{[b-\int_0^1 tg(t)d_qt]-[a-\int_0^1 g(t)d_qt]}{b-\int_0^1 tg(t)d_qt} = \frac{37}{58} \approx 0.637931.$$

Let

$$f(t,u) = \begin{cases} 23, & t\in[0,1], u\in[0,9], \\ 23+600(u-9), & t\in[0,1], u\in[9,10], \\ 623, & t\in[0,1], u\in[10,+\infty). \end{cases}$$

From a direct calculation, we get

$$\begin{aligned} &f(t,u) > M_3 \approx 583.266938 \quad \text{for} \quad t\in[0,1], u\in[10,\tfrac{580}{37}], \\ &f(t,u) < M_2 \approx 36.538326 \quad \text{for} \quad t\in[0,1], u\in[0,9], \\ &f(t,u) > M_1 \approx 21.322041 \quad \text{for} \quad t\in[0,1], u\in[0,0.5]. \end{aligned}$$

In view of Theorem 4.1, we see that the aforementioned problem has at least two positive solutions u_1 and u_2 such that $0.5 < \psi(u_1)$ with $\omega(u_1) < 9$ and $9 < \omega(u_2)$ with $\varphi(u_2) < 10.$

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