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Existence results for fractional *q*-differential equations with integral and multi-point boundary conditions

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ABSTRACT: *This paper concerns a new kind of fractional q-differential equation of arbitrary order by combining a multi-point boundary condition with an integral boundary condition. By solving the equation which is equivalent to the problem we are going to investigate, the Green's functions are obtained. By defining a continuous operator on a Banach space and taking advantage of the cone theory and some fixed-point theorems, the existence of multiple positive solutions for the BVPs is proved based on some properties of Green's functions and under the circumstance that the continuous functions f satisfy certain hypothesis. Finally, examples are provided to illustrate the results.*

KEYWORDS: *fractional q-differential equation; boundary value problem; Green's function; fixed point theorems*

I. INTRODUCTION

Fractional calculus has attracted many researchers' interests because of its wide application in solving practical problems that arise in fields like viscoelasticity, biological science, ecology, aerodynamics, etc. Numerous writings have showed that fractional order differential equations could provide more methods to deal with complex problems in statistical physics and environmental issues. Especially, writers introduced the development history of fractional calculus in [1], and authors in [2] stated some pioneering applications of fractional calculus. For specific applications, see [3, 4] and the references therein. Fractional-order differential equations with boundary value problems sprung up dramatically. Multi-point boundary conditions and integral boundary conditions become hot spots of research among different types of boundary value problems, and the studies in [5–9] are excellent. However, most researchers tend to investigate either integral conditions or multi-point conditions. For instance, the authors explored the fractional-order equation with integral boundary conditions as follows in [9]

$$\begin{cases} D_q^{\alpha}u(t)+h(t)f(t,u(t))=0,\ t\in(0,1), \\ D_q^{j}u(0)=0,\ 0\le j\le n-2,\ u(1)=\mu\int_0^1 g(s)u(s)d_qs, \end{cases} \quad (1.1)$$

where $0<q<1$, D_q^{α} denotes the Riemann-Liouville type fractional *q*-derivative of order α, $\alpha\in(n-1,n]$ and $n\ge 3$ is an integer, $\mu>0$ is a constant, the functions g, h, f are continuous. They devoted themselves to finding the existence of solutions by making use the comparison theorem, monotone iterative technique and lower-upper solution method. In [8], Yang and zhao investigated the following fractional *q*-difference equation by using the Banach contraction principle, Krasnoselskii's fixed point theorem and Schauder fixed point theorem:

$$\begin{cases} {}^cD_q^{\alpha}u(t)=f(t,u(t)),\ t\in[0,1],\ 1<\alpha\le 2, \\ u(0)=\delta u(\sigma),\ a\,{}^cD_q^{\beta}u(\xi_1)+b\,{}^cD_q^{\beta}u(\xi_2)=\sum_{i-1}^{m-2}\gamma_i u(\xi_i),\ 0<\beta\le 1, \end{cases} \quad (1.2)$$

where $0\le\sigma\le\zeta_1\le\xi_1<\cdots<\xi_{m-1}<\zeta_2\le 1\ (i=1,2,\ldots,m-2)$, D_q^{α} represents the Caputo type fractional *q*-derivative of order α. The nonlinear function $f:[0,1]\times R\to R$ is a given continuous function and δ, a, b, $\gamma_i\in R$. In consideration of the fact that integral boundary conditions and multi-point boundary conditions have been investigated in a variety of papers (see [10-12]), in this paper we are dedicated to considering fractional differential equations that contain both the integral boundary condition and the multi-point boundary condition:

$$\begin{cases} D_q^\alpha x(t) + f(t,x(t)) = 0, \\ D_q^i x(0) = 0, \quad i = 0,1,2,\dots n-2, \\ x(1) = \sum_{i=1}^{m-2} \beta_i \int_0^{\eta_i} x(s) d_q s + \sum_{i=1}^{m-2} \gamma_i x(\eta_i), \end{cases} \tag{1.3}$$

where D_q^α represents the standard Riemann-Liouville fractional derivative of order α satisfying $n-1<\alpha\le n$ with $n\ge 3$ and $n\in N^+$. In addition, $\beta_i,\ \gamma_i>0$ and $0<\eta_1<\eta_2<\dots<\eta_{m-2}<1$ with $1\le i\le m-2$, where m is an integer satisfying $m\ge 3$. $f:[0,1]\times R\to R$ is a given continuous function?

To ensure that readers can easily understand the results, the rest of the paper is planned as follows. Section 2 is aimed to recall certain basic definitions and lemmas to obtain the Green's functions. Section 3 is devoted to reviewing Krasnoselkii's fixed point theorem, Schauder type fixed point theorem, Banach's contraction mapping principle and nonlinear alternative for single-valued maps and to applying them to analyze the problem in order to show the main results. In the last section, some examples are given to verify that the results are practical.

II. PRELIMINARIES

Definition 2.1 [13] *Let f be a function defined on* [0,1]. *The fractional q-integral of the Riemann-Liouville type of order* $\alpha\ge 0$ *is* $(I_q^0 f)(t)=f(t)$ *and*

$$I_q^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} f(s) d_q s, \quad \alpha>0, \quad t\in[0,1].$$

Definition 2.2 [14] *The fractional q-derivative of the Riemann-Liouville type of order* α *is*

$$D_q^\alpha f(x) = \begin{cases} I_{q,0^+}^{-\alpha} f(x), \ \alpha<0; \\ f(x), \alpha=0; \\ D_q^{\lceil\alpha\rceil} I_{q,0^+}^{\lceil\alpha\rceil-\alpha} f(x), \ \alpha>0, \end{cases}$$

where $\lceil\alpha\rceil$ denotes the smallest integer greater or equal to α.

Lemma 2.1 [13] *Let* $\alpha>0$, *and n be a positive integer. Then the following equality holds:*

$$I_q^\alpha D_q^n f(t) = D_q^n I_q^\alpha f(t) + \sum_{i=0}^{n-1} \frac{t^{\alpha-n-1}}{\Gamma_q(\alpha+i-n+1)} D_q^i f(0).$$

Lemma 2.2 *Assume that* $h\in L^1[0,1]$, $x\in AC^n[0,1]$ *and* $n-1<\sigma\le n$ *with* $n\ge 3$, *then the solution to the fractional differential equation*

$$D_q^\alpha x(t) + h(t) = 0, \ t\in[0,1] \tag{2.1}$$

with multi-point and integral boundary conditions

$$\begin{cases} D_q^i x(0) = 0, \quad i = 0,1,2,\dots n-2, \\ x(1) = \sum_{i=1}^{m-2} \beta_i \int_0^{\eta_i} x(s) d_q s + \sum_{i=1}^{m-2} \gamma_i x(\eta_i), \end{cases} \tag{2.2}$$

is given by

$$x(t)=\int_0^1 G(t,s)h(s)d_qs+\frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\beta_i\int_0^1 H(\eta_i,s)h(s)d_qs+\frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\gamma_i\int_0^1 G(\eta_i,s)h(s)d_qs,$$

where $\xi=1-\frac{1}{\alpha}\sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha}-\sum_{i=1}^{m-2}\gamma_i\eta_i^{\alpha-1}>0$ *and*

$$G(t,s)=\frac{1}{\Gamma_q(\alpha)}\begin{cases}t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}, & 0\le s\le t\le 1,\\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0\le t\le s\le 1,\end{cases} \tag{2.3}$$

$$H(t,s)=\frac{1}{\Gamma_q(\alpha+1)}\begin{cases}t^{\alpha}(1-s)^{\alpha-1}-(t-s)^{\alpha}, & 0\le s\le t\le 1,\\ t^{\alpha}(1-s)^{\alpha-1}, & 0\le t\le s\le 1.\end{cases} \tag{2.4}$$

Proof By Lemma 2.1, the following equality holds:

$$x(t)=-I_q^{\alpha}h(t)+\sum_{k=1}^{n}C_kt^{\alpha-k}.$$

In view of the boundary conditions, the parameters $C_2=C_3=\cdots=C_n=0$ are concluded and

$$\begin{aligned}x(1)&=-I_q^{\alpha}h(1)+C_1t^{\alpha-1}\\&=\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i}x(s)d_qs+\sum_{i=1}^{m-2}\gamma_ix(\eta_i)\\&=\sum_{i=1}^{m-2}\beta_i\left(-I_q^{\alpha+1}h(\eta_i)+\frac{C_1}{\alpha}\eta_i^{\alpha}\right)+\sum_{i=1}^{m-2}\gamma_i\left(-I_q^{\alpha}h(\eta_i)+C_1\eta_i^{\alpha}\right),\end{aligned}$$

i.e.,

$$\begin{aligned}\left\{1-\frac{1}{\alpha}\sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha}-\sum_{i=1}^{m-2}\gamma_i\eta_i^{\alpha-1}\right\}C_1=&\frac{1}{\Gamma_q(\alpha)}\int_0^1(1-s)^{\alpha-1}h(s)d_qs\\&-\sum_{i=1}^{m-2}\frac{\beta_i}{\Gamma_q(\alpha+1)}\int_0^{\eta_i}(\eta_i-s)^{\alpha}h(s)d_qs\\&-\sum_{i=1}^{m-2}\frac{\gamma_i}{\Gamma_q(\alpha)}\int_0^{\eta_i}(\eta_i-s)^{\alpha-1}h(s)d_qs.\end{aligned}$$

So,

$$\begin{aligned}C_1=\frac{1}{\xi}\Bigg\{&\frac{1}{\Gamma_q(\alpha)}\int_0^1(1-s)^{\alpha-1}h(s)d_qs-\sum_{i=1}^{m-2}\frac{\beta_i}{\Gamma_q(\alpha+1)}\int_0^{\eta_i}(\eta_i-s)^{\alpha}h(s)d_qs\\&-\sum_{i=1}^{m-2}\frac{\gamma_i}{\Gamma_q(\alpha)}\int_0^{\eta_i}(\eta_i-s)^{\alpha-1}h(s)d_qs\Bigg\}.\end{aligned}$$

Hence, the solution is

$$x(t) = -\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} h(s)d_q s + \frac{t^{\alpha-1}}{\xi\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-1)} h(s)d_q s$$

$$-\sum_{i=1}^{m-2}\frac{t^{\alpha-1}\beta_i}{\xi\Gamma_q(\alpha+1)}\int_0^{\eta_i}(\eta_i-qs)^{(\alpha)}h(s)d_q s - \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\frac{\gamma_i}{\Gamma_q(\alpha)}\int_0^{\eta_i}(\eta_i-qs)^{(\alpha-1)}h(s)d_q s$$

$$= -\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} h(s)d_q s + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-1)} h(s)d_q s$$

Therefore,

$$+\frac{(1-\xi)t^{\alpha-1}}{\xi\Gamma_q(\alpha+1)}\int_0^1 (1-qs)^{(\alpha-1)} h(s)d_q s - \sum_{i=1}^{m-2}\frac{t^{\alpha-1}\beta_i}{\xi\Gamma_q(\alpha+1)}\int_0^{\eta_i}(\eta_i-qs)^{(\alpha)}h(s)d_q s$$

$$-\frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\frac{\gamma_i}{\Gamma_q(\alpha)}\int_0^{\eta_i}(\eta_i-qs)^{(\alpha-1)}h(s)d_q s$$

$$= \int_0^1 G(t,s)h(s)d_q s + \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\beta_i\int_0^1 H(\eta_i,s)h(s)d_q s + \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\gamma_i\int_0^1 G(\eta_i,s)h(s)d_q s.$$

we complete the proof. □

Lemma 2.3 *The functions* $G(t,s)$ *and* $H(t,s)$ *obtained in Lemma 2.5 are continuous and nonnegative on* $[0,1]\times[0,1]$ *. It is easy to figure out that* $0 \le G(t,s) \le \frac{1}{\Gamma_q(\alpha)}$ *and* $0 \le H(t,s) \le \frac{1}{\Gamma_q(\alpha+1)}$ *hold for all* $t,s\in[0,1]$.

III. MAIN RESULTS

Based on the lemmas mentioned in the previous section, we define an operator $S : C \to C$ as follows:

$$(Sx)(t) = \int_0^1 G(t,s)f(s,x(s))d_q s + \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\beta_i\int_0^1 H(\eta_i,s)f(s,x(s))d_q s$$
$$+\frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\gamma_i\int_0^1 G(\eta_i,s)f(s,x(s))d_q s, \qquad (3.1)$$

where $C = C([0,1], R)$ denotes the Banach space of all continuous functions defined on $[0,1]$ that are mapped into R with the norm defined as $\|x\| = \sup_{t\in[0,1]}|x(t)|$. And G and H are provided in Lemma 2.6.

Solutions to the problem exist if and only if the operator S has fixed points. In order to make the analysis clear, we introduce some fixed-point theorems that play a role in our proof.

Lemma 3.1 (Krasnoselskii [19]) *Let* Q *be a closed, convex, bounded and nonempty subset of a Banach space* Y. *Let* ψ_1, ψ_2 *be operators such that*

(i) $\psi_1 v_1 + \psi_2 v_2 \in Q$ *whenever* $v_1, v_2 \in Q$;

(ii) ψ_1 *is compact and continuous;*

(iii) ψ_2 *is a contraction mapping.*

Then there exists $v \in Q$ *such that* $v = \psi_1 v_1 + \psi_2 v_2$.

Lemma 3.2 ([14]) *Let X be a Banach space. Assume that* $T : X \to X$ *is a completely continuous operator and the set* $V = \{u \in X : u = \varepsilon Tu, 0<\varepsilon<1\}$ *is bounded. Then* T *has a* *fixed point in* X .

Lemma 3.3 ([15]) *Let* E *be a Banach space,* E_1 *be a closed, convex subset of* E , V *be an open subset of* E_1 *and* $0 \in V$. *Suppose that* $U : \overline{V} \to E_1$ *is a continuous, compact (that is,* $U(\overline{V})$ *is a relatively compact subset of* E_1*) map. Then either*
(i) U *has a fixed point in* V *, or*
(ii) there are $x \in \partial V$ *(the boundary of* V *in* E_1 *and* $k \in (0,1)$ *with* $x = kU(x)$.

For convenience, we set some notations:

$$\alpha_1 = \frac{1}{\Gamma_q(\alpha)} + \frac{1}{\xi}\sum_{i=1}^{m-2}\beta_i \frac{(\alpha+1)\eta_i^{\alpha} - \alpha\eta_i^{\alpha+1}}{\alpha(\alpha+1)\Gamma_q(\alpha+1)} + \frac{1}{\xi}\sum_{i=1}^{m-2}\gamma_i \frac{\eta_i^{\alpha-1} - \eta_i^{\alpha}}{\Gamma_q(\alpha+1)},$$

$$\alpha_2 = \alpha_1 - \frac{1}{\Gamma_q(\alpha)}. \tag{3.2}$$

Theorem 3.4 *Let* $f : [0,1] \times R \to R$ *be a continuous function that satisfies the conditions:*
(*H*1) $\left| f(t,x) - f(t,y) \right| \le l\left| x - y \right|$ *for all* $t \in [0,1]$ *and* $x, y \in R$ *;*
(*H*2) $\left| f(t,x) \right| \le \omega(t)$ *for all* $(t,x) \in [0,1] \times R$ *and* $\omega \in C([0,1], R^{+})$ *,*
then BVP (1.3) *has at least one solution on* $[0,1]$ *when* $l\alpha_2 \le 1$ *with* α_2 *given in* (3.2).

Proof Define a set $B_\rho = \{x \in C : \|x\| \le \rho\}$, where $\rho \ge \|\omega\|\alpha_1$ with α_1 defined in (3.2) and $\|\omega\| = \sup_{t\in[0,1]}|\omega(t)|$. Let the operators S_1 and S_2 on B_ρ be defined as

$$(S_1x)(t) = \int_0^1 G(t,s) f(s,x(s)) d_q s,$$

$$(S_2x)(t) = \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\beta_i \int_0^1 H(\eta_i,s) f(s,x(s)) d_q s + \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\gamma_i \int_0^1 G(\eta_i,s) f(s,x(s)) d_q s.$$

It is easy to understand $\|S_1x + S_2y\| \le \|\omega\|\alpha_1$ for any $x, y \in B_\rho$ that means $S_1x + S_2y \in B_\rho$.
By assumption (*H*1),

$$\begin{aligned}\|S_2x - S_2y\| &= \sup_{t\in[0,1]}|S_2x - S_2y| \\ &\le \sup_{t\in[0,1]} \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\beta_i \int_0^1 H(\eta_i,s)\left| f(s,x) - f(s,y)\right| d_q s \\ &\quad + \sup_{t\in[0,1]} \frac{1}{\xi}\sum_{i=1}^{m-2}\beta_i \int_0^1 G(\eta_i,s)\left| f(s,x) - f(s,y)\right| d_q s \\ &\le \frac{l}{\xi}\left\{\sum_{i=1}^{m-2}\beta_i \frac{(\alpha+1)\eta_i^{\alpha} - \alpha\eta_i^{\alpha+1}}{\alpha(\alpha+1)\Gamma_q(\alpha+1)} + \sum_{i=1}^{m-2}\gamma_i \frac{\eta_i^{\alpha-1} - \eta_i^{\alpha}}{\Gamma_q(\alpha+1)}\right\}\|x - y\| \\ &\le l\alpha_2\|x - y\|.\end{aligned}$$

The operator S_2 is a contraction because of $l\alpha_2 \le 1$. As we all know, S_1 is a continuous result from the continuity of f . Moreover,

$$\|S_1x\| = \sup_{t\in[0,1]}\left\{\int_0^1 G(t,s) f(s,x(s)) d_q s\right\} \le \sup_{t\in[0,1]}\left\{\int_0^1 \frac{|f(s,x(s))|}{\Gamma_q(\alpha)} d_q s\right\} \le \frac{\|\omega\|}{\Gamma_q(\alpha)},$$

which implies that S_1 is uniformly bounded on B_ρ . Apart from that, the following inequalities hold with $\sup_{(t,x)\in[0,1]\times B_\rho}|f(t,x)| = f_m < +\infty$ and $0 < t_1 < t_2 < 1$:

$$\left|(S_2x)(t_2)-(S_2x)(t_1)\right| = \left|\int_0^1 G(t_2,s)f(s,x(s))d_qs - \int_0^1 G(t_1,s)f(s,x(s))d_qs\right|$$
$$\leq \int_0^1 [G(t_2,s)-G(t_1,s)]f(s,x(s))d_qs$$
$$\leq \frac{f_m}{\Gamma_q(\alpha+1)}\left[(t_2^{\alpha-1}-t_1^{\alpha-1})+(t_2^{\alpha}-t_1^{\alpha})\right].$$

The operator S_1 is compact due to the Arzela-Ascoli theorem. Since three conditions are satisfied, BVP (1.3) has at least one solution on $[0,1]$ by application of Krasnoselskii's fixed point theorem. □

Theorem 3.5 *Assume that there exists a constant* L *such that* $|f(t,x)| \leq L$ *for any* $t\in[0,1]$ *and* $x \in C[0,1]$. *Then there exists at least one solution to BVP* (1.3).

Proof Firstly, we set out to verify that the operator S given in (3.1) is completely continuous. Define a bounded set $U \subset C([0,1],R^+)$, then $|(Sx)(t)| \leq L\alpha_1$ holds when we take $x \in U$. On the top of that,

$$\left|(Sx)(t_2)-(Sy)(t_1)\right| = \left|\int_0^1 [G(t_2,s)-G(t_1,s)]f(s,x(s))d_qs\right.$$
$$+\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\beta_i\int_0^1 H(\eta_i,s)f(s,x(s))d_qs$$
$$\left.+\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\gamma_i\int_0^1 G(\eta_i,s)f(s,x(s))d_qs\right|$$
$$\leq L\left\{\frac{(t_2^{\alpha-1}-t_1^{\alpha-1})+(t_2^{\alpha}-t_1^{\alpha})}{\Gamma_q(\alpha+1)}+\alpha_2(t_2^{\alpha-1}-t_1^{\alpha-1})\right\}.$$

Hence, S is equicontinuous on $[0,1]$ in view of t^{σ} and $t^{\sigma-1}$ is equicontinuous on $[0,1]$. The operator S is deduced to be completely continuous by the Arzela
-Ascoli theorem along with the continuity of S decided by f.
Secondly, we consider the set $V=\{x\in C: x=\lambda Sx, 0<\lambda<1\}$ and prove that V is bounded. In fact, for each $x \in V$ and $t\in[0,1]$,

$$\|x\| = \|\lambda(Sx)(t)\|$$
$$\leq L\left\{\frac{1}{\Gamma_q(\alpha)}+\frac{1}{\xi}\sum_{i=1}^{m-2}\beta_i\frac{(\alpha+1)\eta_i^{\alpha}-\alpha\eta_i^{\alpha+1}}{\alpha(\alpha+1)\Gamma_q(\alpha+1)}+\frac{1}{\xi}\sum_{i=1}^{m-2}\gamma_i\frac{\eta_i^{\alpha-1}-\eta_i^{\alpha}}{\Gamma_q(\alpha+1)}\right\}$$
$$= L\alpha_1$$

Consequently, the set V is bounded by definition.
Finally, we conclude that BVP (1.3) has at least one solution according to Lemma 3.5 and the proof is completed.

Theorem 3.6 Assume that $f:[0,1]\times R \to R$ is a continuous function and satisfies condition (H_1) with $l\alpha_1 \leq 1$, where α_1 is defined in (3.2). Then the BVP has a unique solution on $[0,1]$.

Proof Let $P_r=\{x\in C: \|x\|\leq r\}$ be a bounded set. To show $SP_r \subset P_r$ with the operator S defined in (3.1), $|f(s,x(s))| \leq lr+\delta$ holds when we take $x \in P_r$ for $t\in[0,1]$ and with the condition provided by $\sup_{t\in[0,1]}|f(t,0)|=\delta$ and $r \geq \frac{\delta\alpha_1}{1-l\alpha_1}$. Beyond that,

$$\|(Sx)\| \le \sup_{t\in[0,1]}\left\{\int_0^1 G(t,s)|f(s,x(s))|d_q s + \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\beta_i\int_0^1 H(\eta_i,s)|f(s,x(s))|d_q s \right.$$
$$\left. + \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\gamma_i\int_0^1 G(\eta_i,s)|f(s,x(s))|d_q s\right\},$$
$$\le (lr+\delta)\alpha_1 \le r.$$

$SP_r \subset P_r$ is strictly proved. Next, choosing $x, y \in C$ with $t \in [0,1]$, we get

$$\|Sx - Sy\| \le \sup_{t\in[0,1]}\left\{\int_0^1 G(t,s)|f(s,x(s)) - f(s,y(s))|d_q s \right.$$
$$+ \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\beta_i\int_0^1 H(\eta_i,s)|f(s,x(s)) - f(s,y(s))|d_q s$$
$$\left. + \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\gamma_i\int_0^1 G(\eta_i,s)|f(s,x(s)) - f(s,y(s))|d_q s\right\}$$
$$\le l\alpha_1\|x-y\|.$$

The operator S is a contraction with the assumption $l\alpha_1 < 1$. Therefore, BVP (1.3) has a unique solution by Banach's contraction mapping principle. □

Theorem 3.7 *Let* $f:[0,1]\times R \to R$ *be a continuous function, and assume that*

(H_3) *there exist a function* $p \in C([0,1], R^+)$ *and a nondecreasing function* $q : R^+ \to R^+$ *such that* $|f(t,x)| \le p(t)q(\|x\|)$ for all $(t,x) \in [0,1]\times R$;

(H_4) *there exists a constant* $N > 0$ *such that*

$$N\left\{q(v)\|p\|\left\{\frac{1}{\Gamma_q(\alpha)} + \frac{1}{\xi}\sum_{i=1}^{m-2}\beta_i\frac{(\alpha+1)\eta_i^{\alpha} - \alpha\eta_i^{\alpha+1}}{\alpha(\alpha+1)\Gamma_q(\alpha+1)} + \frac{1}{\xi}\sum_{i=1}^{m-2}\gamma_i\frac{\eta_i^{\alpha-1} - \eta_i^{\alpha}}{\Gamma_q(\alpha+1)}\right\}\right\}^{-1} \le 1.$$

Then BVP (1.3) *has at least one solution on* [0,1].

Proof The first step is to show that the operator S given in (3.1) maps bounded sets into bounded sets in C. Let v be a positive number and $B_v = \{x \in C : \|x\| \le v\}$ be a bounded set in C. For each $x \in B_v$ and by (H_3), the following equalities are obtained:

$$|(Sx)(t)| \le \int_0^1 G(t,s)|f(s,x(s))|d_q s + \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\beta_i\int_0^1 H(\eta_i,s)|f(s,x(s))|d_q s$$
$$+ \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\gamma_i\int_0^1 G(\eta_i,s)|f(s,x(s))|d_q s$$
$$\le q(v)\|p\|\left\{\frac{1}{\Gamma_q(\alpha)} + \frac{1}{\xi}\sum_{i=1}^{m-2}\beta_i\frac{(\alpha+1)\eta_i^{\alpha} - \alpha\eta_i^{\alpha+1}}{\alpha(\alpha+1)\Gamma_q(\alpha+1)} + \frac{1}{\xi}\sum_{i=1}^{m-2}\gamma_i\frac{\eta_i^{\alpha-1} - \eta_i^{\alpha}}{\Gamma_q(\alpha+1)}\right\}$$
$$= q(v)\|p\|\alpha_1.$$

The next step is to verify that the operator S maps bounded sets into equicontinuous sets of C.
Choose $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and take $x \in B_v$,

$$
\begin{aligned}
|(Sx)t_2-(Sy)t_1| &= \Big|\int_0^1[G(t_2,s)-G(t_1,s)]f(s,x(s))d_qs \\
&\quad+\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\beta_i\int_0^1 H(\eta_i,s)f(s,x(s))d_qs \\
&\quad+\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\gamma_i\int_0^1 G(\eta_i,s)f(s,x(s))d_qs\Big| \\
&\le q(v)\|p\|\left\{\frac{(t_2^{\alpha-1}-t_1^{\alpha-1})+(t_2^{\alpha}-t_1^{\alpha})}{\Gamma_q(\alpha+1)}+\alpha_2(t_2^{\alpha-1}-t_1^{\alpha-1})\right\}
\end{aligned}
$$

By the Arzela-Ascoli theorem, the operator S is completely continuous since the right-hand side tends to zero independent of $x\in B_v$ as $t_2\to t_1$. Let x be a solution of BVP (1.3), then for $\mu\in(0,1)$ and by the same method applied to show the boundedness of S, the results hold:

$$
\begin{aligned}
|x(t)| &= |\mu(Sx)(t)| \\
&\le q(\|x\|)\|p\|\left\{\frac{1}{\Gamma_q(\alpha)}+\frac{1}{\xi}\sum_{i=1}^{m-2}\beta_i\frac{(\alpha+1)\eta_i^{\alpha}-\alpha\eta_i^{\alpha+1}}{\alpha(\alpha+1)\Gamma_q(\alpha+1)}+\frac{1}{\xi}\sum_{i=1}^{m-2}\gamma_i\frac{\eta_i^{\alpha-1}-\eta_i^{\alpha}}{\Gamma_q(\alpha+1)}\right\},
\end{aligned}
$$

i.e.,

$$
\|x\|\left\{q(v)\|p\|\left\{\frac{1}{\Gamma_q(\alpha)}+\frac{1}{\xi}\sum_{i=1}^{m-2}\beta_i\frac{(\alpha+1)\eta_i^{\alpha}-\alpha\eta_i^{\alpha+1}}{\alpha(\alpha+1)\Gamma_q(\alpha+1)}+\frac{1}{\xi}\sum_{i=1}^{m-2}\gamma_i\frac{\eta_i^{\alpha-1}-\eta_i^{\alpha}}{\Gamma_q(\alpha+1)}\right\}\right\}^{-1}\le 1.
$$

The final step is to select a set $Q=\{x\in C:\|x\|<N+1\}$ in order to take (H_4) into consideration where there exists N such that $\|x\|\ne N$. Even though $S:\overline{Q}\to C$ is completely continuous can be verified, there is no $x\in\partial N$ that can satisfy $x=\mu S(x)$ for $\mu\in(0,1)$ decided by the selection of N.

Above all, we complete the proof that the operator S has a fixed point in Q which is a solution to BVP (1.3).

□

IV. EXAMPLE

Example 4.1 Consider the fractional differential equations with boundary value as follows:

$$
\begin{cases}
D_q^{\frac{7}{2}}x(t)+t\sin x+e^{-t}\tan^{-1}x+\dfrac{1}{2}=0,\ \ t\in[0,1],\\[2mm]
D_qx(0)=0,\quad D_q^2x(0)=0,\\[2mm]
x(1)=\dfrac{5}{2}\int_0^{\frac{1}{4}}x(s)d_qs+4\int_0^{\frac{1}{3}}x(s)d_qs+\dfrac{3}{2}x(\dfrac{1}{4})+2x(\dfrac{1}{3}).
\end{cases}
\tag{4.1}
$$

From the equation above, it is clear that $\alpha=\frac{7}{2}$, $m=2$, $\beta_1=\frac{5}{2}$, $\beta_2=4$, $\gamma_1=\frac{3}{2}$, $\gamma_2=2$, $\eta_1=\frac{1}{4}$, $\eta_2=\frac{1}{3}$. Consequently, we can get $\xi=0.026$, $\alpha_1=0.7918$, $\alpha_2=0.4785$ by computation.

Since $f(t,x)=t\sin x+e^{-t}\cos x+\frac{1}{2}$,

$$
\begin{aligned}
|f(t,x)-f(t,y)| &= |t\sin x-t\sin y+e^{-t}\tan^{-1}x-e^{-t}\tan^{-1}y| \\
&\le t|\sin x-\sin y|+e^{-t}|\tan^{-1}x-\tan^{-1}y| \\
&\le 2|x-y|,
\end{aligned}
$$

that implies $l = 2$ and both $l\alpha_1 < 1$ and $l\alpha_2 < 1$ hold. Problem (4.1) has a unique solution on [0,1] because all the conditions of Theorem 3.6 are satisfied.

V. CONCLUSION

We have proved the existence of solutions for fractional differential equations with integral and multi-point boundary conditions. The problem is issued by applying some fixed point theorems and the properties of Green's function. We also provide examples to make our results clear.

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