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Existence of solutions for nabla $Caputo$ left fractional boundary value problems with a $p-Laplacian$ operator

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ABSTRACT : *In this paper, we investigate the existence of solutions for nabla $Caputo$ Left fractional boundary value problems with a $p-Laplacian$ operator. Under the nonlinear growth conditions, the existence result is established by using $Schaefer's$ fixed point theorem. An example is added to illustrate the main result.*

KEYWORDS: *fractional boundary value problem; $p-Laplacian$ operator; $Schaefer's$ fixed point theorem*

I. INTRODUCTION

For any real number β. Let $N_\beta=\{\beta,\beta+1,\beta+2,\ldots\}$, It is also worth noting that, in what follows, we appeal to the convention that $\sum_{s=k}^{k-1}u(s)=0$ for any $k\in N_\beta$, where u is a function defined on N_β.

In this paper, we investigate the existence of solutions for the following discrete fractional boundary value problem with $p-Laplacian$:

$$\begin{cases} {}^C\nabla_a^\beta\left[\phi_p\left({}^C\nabla_a^\alpha u(t)\right)\right]=f\left(t,u(t)\right) \quad , \quad t\in[a,b]_{N_a}, \\ {}^C\nabla_a^\alpha u(t)\big|_{t=a}+{}^C\nabla_a^\alpha u(t)\big|_{t=b}=0, \\ u(a)+u(b)=0. \end{cases} \tag{1.1}$$

Where $0<\alpha,\beta\le 1, 1<\alpha+\beta\le 2$, $b\in N_1$, ${}^C\nabla_a^\beta$ denote the nabla $Caputo$ left fractional difference of $\beta-order$, $f:[a+\alpha+\beta-2,a+\alpha+\beta+b]_{N_{a+\alpha+\beta-2}}\times R\to R$ is a continuous function and ϕ_p is the $p-Laplacian$ operator, that is $\phi_p(u)=|u|^{p-2}u$, $p>1$. Obviously, ϕ_p is invertible and its inverse operator is ϕ_p, where $q>1$ is a constant such that $\frac{1}{p}+\frac{1}{q}=1$.

Preliminaries : In this section, we first present here some necessary basic definitions on nabla $Caputo$ left fractional calculus.

Definition 2.1[7] (i) For a natural number m, the fractional of t is defined by

$$t^{\overline{m}}=\prod_{k=0}^{m-1}(t+k), t^{\overline{0}}=1$$

(ii) For any real number the α rising function is defined by

$$t^{\overline{\alpha}}=\frac{\Gamma(t+\alpha)}{\Gamma(t)}, t\in R\setminus\{\cdots,-2,-1,0\}, 0^\alpha=0.$$

Definition 2.2[6] Let $\rho(s)=s-1$, the (nabla) left fractional sum of order $\alpha>0$ (starting from a) is defined by

$$\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} f(s), t \in N_{a+1}$$

Definition 2.3[6] Let $\alpha > 0, \alpha \notin N$, $\rho(s) = s-1$, the nabla $\alpha - order\, Caputo$ left fractional difference of a function f defined on N_a and some points before a is defined by

$$\begin{aligned} {}^{C}\nabla_a^{\alpha} f(t) &\triangleq \nabla_a^{-(n-\alpha)} \nabla^n f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \sum_{s=a+1}^{t-(n-\alpha)} (t-\rho(s))^{\overline{n-\alpha-1}} \nabla^n f(s). \end{aligned}$$

If $\alpha = n \in N$, then

$${}^{C}\nabla_a^{\alpha} f(t) \triangleq \nabla^n f(t)$$

Definition 2.4[6] Let $\rho(s) = s-1$, the (nabla) left fractional difference of order $\alpha > 0$ (starting from a) is defined by

$$\begin{aligned} \nabla_a^{\alpha} f(t) &= \nabla^n \nabla_a^{-(n-\alpha)} f(t) \\ &= \frac{\nabla^n}{\Gamma(n-\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{n-\alpha-1}} f(s), t \in N_{a+1}. \end{aligned}$$

Lemma2.1[6] Assume that $\nu > 0$ and f is defined on N_a. Then

$${}^{C}\nabla_a^{\alpha} f(t) = \nabla_a^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\overline{k-\alpha}}}{\Gamma(k-\alpha+1)} \nabla^k f(a).$$

Lemma2.2[6] Assume that $\alpha > 0$, $n = [\alpha]+1$ and f is defined on suitable domains N_a. Then

$$\nabla_a^{-\alpha}\, {}^{C}\nabla_a^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\overline{k}}}{k!} \nabla^k f(a).$$

II. EXISTENCE RESULT

In this section, we will establish the existence of at least one solution for problem (1.1). To accomplish this, we first state and prove the following result which is of particular importance in what follow.

Lemma 3.1 Let $h : [\alpha+\beta-1, \alpha+\beta-1+b] N_{\alpha+\beta-1} \to R$ be given, the problem

$$\begin{cases} {}^{C}\nabla_a^{\beta}\left[\phi_p\left({}^{C}\nabla_a^{\alpha} u(t)\right)\right] = h(t), \\ {}^{C}\nabla_a^{\alpha} u(t)\big|_{t=a} + {}^{C}\nabla_a^{\alpha} u(t)\big|_{t=b} = 0, \\ u(a) + u(b) = 0. \end{cases} \qquad (3.1)$$

has the unique solution

$$\begin{aligned} u(t) = &\left[\frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} - \frac{1}{2\Gamma(\alpha)} \sum_{s=a+1}^{b} (b-\rho(s))^{\overline{\alpha-1}} \right] \\ &\phi_q\left[\frac{1}{\Gamma(\beta)} \sum_{\tau=a+1}^{s} (s-\rho(\tau))^{\overline{\beta-1}} h(\tau) - \frac{1}{2\Gamma(\beta)} \sum_{\tau=a+1}^{b} (b-\rho(\tau))^{\overline{\beta-1}} h(\tau) \right] \end{aligned}$$

Proof: by Lemma2.3 implies that

$$\phi_p\left({}^C\nabla_a^\alpha u(t)\right)=\frac{1}{\Gamma(\beta)}\sum_{s=a+1}^{t}\left(t-\rho(s)\right)^{\overline{\beta-1}}h(s)+c_0$$

for some $c_0\in R,\ t\in[\beta-1,b+\beta]_{N_{\beta-1}}$.

From the boundary condition ${}^C\nabla_a^\alpha u(t)\big|_{t=a}+{}^C\nabla_a^\alpha u(t)\big|_{t=b}=0$, one has

$$c_0=-\frac{1}{2\Gamma(\beta)}\sum_{s=a+1}^{b}(b-s+1)^{\overline{\beta-1}}h(s),$$

Then

$${}^C\nabla_a^\alpha u(t)=\phi_q\left[\frac{1}{\Gamma(\beta)}\sum_{s=a+1}^{t}\left(t-\rho(s)\right)^{\overline{\beta-1}}h(s)-\frac{1}{2\Gamma(\beta)}\sum_{s=a+1}^{b}\left(b-s+1\right)^{\overline{\beta-1}}h(s)\right]$$

Therefore, we have

$$u(t)=c_1+\frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}\left(t-\rho(s)\right)^{\overline{\alpha-1}}\phi_q\left[\frac{1}{\Gamma(\beta)}\sum_{\tau=a+1}^{s}\left(s-\rho(\tau)\right)^{\overline{\beta-1}}h(\tau)-\frac{1}{2\Gamma(\beta)}\sum_{\tau=a+1}^{b}\left(b-\tau+1\right)^{\overline{\beta-1}}h(\tau)\right] \tag{3.2}$$

where $c_1\in R,\ t\in[\alpha+\beta-2,b+\alpha+\beta]_{N_{\alpha+\beta-2}}$.

From the boundary condition $u(a)+u(b)=0$, we have

Lemma 3.2[5] ($Schaefer's$ fixed point theorem) Let T be a continuous and compact mapping of a Banach space E into itself such that the set

$S=\{x\in E | x=\lambda Tx,\ for\,some\,\lambda\in(0,1)\}$ is bounded . Then T has a fixed point.

In order to use $Schaefer's$ fixed point theorem to solve the problem (1.1) , define the operator $F:C[a,b]_{N_a}\to C[a,b]_{N_a}$ by

$$Fu(t)=$$

$$\frac{-1}{2\Gamma(\alpha)}\sum_{s=a+1}^{b}\left(b-\rho(s)\right)^{\overline{\alpha-1}}\phi_q\left[\frac{1}{\Gamma(\beta)}\sum_{\tau=a+1}^{s}\left(s-\rho(\tau)\right)^{\overline{\beta-1}}\Psi u(\tau)-\frac{1}{2\Gamma(\beta)}\sum_{\tau=a+1}^{b}\left(b-\rho(\tau)\right)^{\overline{\beta-1}}\Psi u(\tau)\right]$$

$$+\frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}\left(t-\rho(s)\right)^{\overline{\alpha-1}}\phi_q\left[\frac{1}{\Gamma(\beta)}\sum_{\tau=a+1}^{s}\left(s-\rho(\tau)\right)^{\overline{\beta-1}}\Psi u(\tau)-\frac{1}{2\Gamma(\beta)}\sum_{\tau=a+1}^{b}\left(b-\rho(\tau)\right)^{\overline{\beta-1}}\Psi u(\tau)\right]$$

where $C[a,b]_{N_a}$ denotes the Banach space of all functions $u:[a,b]_{N_a}\to R$ with the norm $\|u\|=\max\{|u(t)|:t\in[a,b]_{N_a}\}$, and define the operator $\Psi:C[a,b]_{N_a}\to C[a,b]_{N_a}$ by

$$\Psi u(t)=f\left(t,u(t)\right),\quad \forall t\in[a,b]_{N_a}.$$

It is easy to verify that the operator F is well defined , and the fixed points of the operator F are solutions of problem (1.1).

Now, the main result is stated as follows,

Theorem3.1 Assume that there exist nonnegative function $e, g \in C[a,b]_{N_a}$, such that

$$\left| f(t,u)\right| \le e(t)+g(t)|u|^{p-1}, \ \forall t\in [a,b]_{N_a},\ u\in R. \qquad (3.3)$$

Then the problem (1.1) has a least one solution , provided that

$$\left(\frac{3}{2}\right)^{q}\|g\|^{q-1}\left(\frac{\Gamma(b-a+\beta)}{\Gamma(\beta+1)\Gamma(b-a)}\right)^{q-1}\frac{\Gamma(b-a+\alpha)}{\Gamma(\alpha+1)\Gamma(b-a)}<1. \qquad (3.4)$$

Proof: The proof will be divided into the following two steps.

Step 1: $F: C[a,b]_{N_a} \to C[a,b]_{N_a}$ is completely continuous.

At first, considering the continuity of f , we have F is continuous. Furthermore, it is not difficult to verify that F maps bounded sets into bounded sets and equi-continuous sets. Therefore, according to $Arzela-Ascoli$ theorem, we know that F is a compact operator.

Step 2: F a priori bounds.

Set $S=\{u\in C[a,b]_{N_a} \,|\, u=\lambda Fu,\ \lambda\in(0,1)\}$

Now, it remains to show that the set S is bounded.

For any $u\in S$, there exists a $\lambda\in(0,1)$ such that $u(t)=\lambda Fu(t)$. So, by(3.3),

Lemma2.4 and the monotonicity of $s^{\overline{\ell}}, \ell\in(0,1]$, we can obtain that

$$|u(t)|=\lambda|Fu(t)|\le$$

$$\frac{-1}{2\Gamma(\alpha)}\sum_{s=a+1}^{b}(b-\rho(s))^{\overline{\alpha-1}}\phi_q\left[\frac{1}{\Gamma(\beta)}\sum_{\tau=a+1}^{s}(s-\rho(\tau))^{\overline{\beta-1}}\Psi u(\tau)-\frac{1}{2\Gamma(\beta)}\sum_{\tau=a+1}^{b}(b-\rho(\tau))^{\overline{\beta-1}}\Psi u(\tau)\right]$$

$$+\frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\phi_q\left[\frac{1}{\Gamma(\beta)}\sum_{\tau=a+1}^{s}(s-\rho(\tau))^{\overline{\beta-1}}\Psi u(\tau)-\frac{1}{2\Gamma(\beta)}\sum_{\tau=a+1}^{b}(b-\rho(\tau))^{\overline{\beta-1}}\Psi u(\tau)\right]$$

$$=\frac{-1}{2\Gamma(\alpha)}\sum_{s=a+1}^{b}(b-\rho(s))^{\overline{\alpha-1}}\phi_q\left[\frac{\|e\|+\|g\|\|u\|^{p-1}(s-a)^{\overline{\beta}}}{\Gamma(\beta+1)}+\frac{\|e\|+\|g\|\|u\|^{p-1}(b-a)^{\overline{\beta}}}{2\Gamma(\beta+1)}\right]$$

$$+\frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\phi_q\left[\frac{\|e\|+\|g\|\|u\|^{p-1}(s-a)^{\overline{\beta}}}{\Gamma(\beta+1)}+\frac{\|e\|+\|g\|\|u\|^{p-1}(b-a)^{\overline{\beta}}}{2\Gamma(\beta+1)}\right]$$

$$\le\phi_q\left[\frac{\|e\|+\|g\|\|u\|^{p-1}(s-a)^{\overline{\beta}}}{\Gamma(\beta+1)}+\frac{\|e\|+\|g\|\|u\|^{p-1}(b-a)^{\overline{\beta}}}{2\Gamma(\beta+1)}\right]$$

$$\left[\frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}-\frac{1}{2\Gamma(\alpha)}\sum_{s=a+1}^{b}(b-\rho(s))^{\overline{\alpha-1}}\right]$$

$$\le \phi_q\left[\frac{\left(\|e\|+\|g\|\|u\|^{p-1}\right)(s-a)^{\bar{\beta}}}{\Gamma(\beta+1)}+\frac{\left(\|e\|+\|g\|\|u\|^{p-1}\right)(b-a)^{\bar{\beta}}}{2\Gamma(\beta+1)}\right]\left[\frac{(t-a)^{\bar{\alpha}}}{\Gamma(\alpha+1)}+\frac{(b-a)^{\bar{\alpha}}}{2\Gamma(\alpha+1)}\right]$$

$$\le \frac{3(b-a)^{\bar{\alpha}}}{2\Gamma(\alpha+1)}\phi_q\left[\frac{3(b-a)^{\bar{\beta}}}{2\Gamma(\beta+1)}\left(\|e\|+\|g\|\|u\|^{p-1}\right)\right]$$

$$\le \left(\frac{3}{2}\right)^q\left(\frac{(b-a)^{\bar{\beta}}}{\Gamma(\beta+1)}\right)^{q-1}\frac{(b-a)^{\bar{\alpha}}}{\Gamma(\alpha+1)}\left(\|e\|+\|g\|\|u\|^{p-1}\right)^{q-1}$$

$$=\left(\frac{3}{2}\right)^q\left(\frac{\Gamma(b-a+\beta)}{\Gamma(\beta+1)\Gamma(b-a)}\right)^{q-1}\frac{\Gamma(b-a+\beta)}{\Gamma(\alpha+1)\Gamma(b-a)}\left(\|e\|+\|g\|\|u\|^{p-1}\right)^{q-1} \tag{3.5}$$

So, we consider the following two cases: (1) $\|g\|=0$ or (2) $\|g\|\neq 0$

Case 1: Suppose that $\|g\|=0$, then it is evident that the set S is bounded from (3.5)

Case 2: Suppose that $\|g\|\neq 0$, it also follows from (3.5) that

$$\|u\|^{p-1}\le\left\{\left(\frac{3}{2}\right)^q\|g\|^{q-1}\left(\frac{\Gamma(b-a+\beta)}{\Gamma(\beta+1)\Gamma(b-a)}\right)^{q-1}\frac{\Gamma(b-a+\alpha)}{\Gamma(\alpha+1)\Gamma(b-a)}\right\}^{q-1}\left(\frac{\|e\|}{\|g\|}+\|u\|^{p-1}\right) \tag{3.6}$$

By virtue of (3.4) and (3.6), it is obvious that there exists a constant $M>0$ such that $\|u\|\le M$.

Consequently, in both Case 1 and Case 2, we have proved that the set S is bounded.

Then, we can see that F satisfies all conditions of $Schaefer's$ fixed point theorem. Thus, we approach a conclusion that F has at least one fixed point which is the solution of problem (1.1). The proof is complete.

III. AN ILLUSTRATIVE EXAMPLE

In order to illustrate the main result, we give the following example.

Example 4.1 Consider the following discrete fractional boundary value problem:

$$\begin{cases} {}^C\nabla_0^{\frac{2}{3}}\left[\phi_{\frac{3}{2}}\left({}^C\nabla_0^{\frac{3}{4}}u(t)\right)\right]=\sin\left(t+\frac{1}{6}\right) & t\in[0,2]_{N_0},\ u\in R,\\ {}^C\nabla_0^{\frac{3}{4}}u(t)\big|_{t=0}+{}^C\nabla_0^{\frac{3}{4}}u(t)\big|_{t=2}=0, & \\ u(0)+u(2)=0. & \end{cases} \tag{4.1}$$

Corresponding to problem (1.1) , we have $q=3,\ p=\frac{3}{2},\ \alpha=\frac{3}{4},\ \beta=\frac{2}{3},$

$$a=0,\ b=2,\ f(t,u(t))=\sin t,\ t\in\left[\frac{1}{6},\frac{13}{6}\right]_{N_{\frac{1}{6}}},\ u\in R.$$

Choose $e(t)=1,\ g(t)=\frac{1}{10}$. By a simple calculation , we can obtain

$$\left(\frac{3}{2}\right)^3\left(\frac{1}{10}\right)^2\left(\frac{\Gamma\left(2+\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}+1\right)\Gamma(2)}\right)^2\frac{\Gamma\left(2+\frac{3}{4}\right)}{\Gamma\left(\frac{3}{4}+1\right)\Gamma(2)}=0.1640625<1.$$

Obviously , problem (4.1) satisfies all assumptions of Theorem 3.1 . Hence , we can conclude that problem (4.1) has at least one solution .

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